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ALGEBRAIC METHODS IN SYSTEM THEORY

by R.W. Brockett, J.C. Willems and A.S. Willsky

HARVARD UNIVERSITY
Division of Engineering and Applied Physics
Cambridge, Massachusetts 02138

prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ALGEBRAIC DECOMPOSITION METHODS FOR NONLINEAR SYSTEMS*

Roger W. Brockett
Division of Engineering and Applied Physics
Harvard University
Cambridge, Massachusetts

Abstract

Elegant algebraic theories for decomposing dynamical systems into elementary pieces have existed for some time in the areas of finite automata and linear systems. In contemporary physics, algebraic ideas, especially Lie algebras and Lie groups are used extensively to reveal and explain structure. This paper is an informal survey bringing together some of the important view points found in these areas. We find that although it is usually helpful, in many cases linearity is not crucial.

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1. Introduction

The main point of this paper is that the utility of the mapping semigroup discussed by Myhill [1] in the study of the structure of dynamical input-output models is by no means limited to the finite state, discrete time case. In many different settings it is the algebraic structures which one can give this set of maps which reveal the possibilities for decomposing the system. The type of decomposition one seeks will, of course, depend on the structure one wants for the subsystems. The standard structure theorems of algebra provide the tools. The class of systems we treat are not characterized by linearity but instead they are characterized by the algebraic structures which the mapping semigroup admits.

To be sure, the general principles on which this paper is based are implicit in the literature. However, they do not stand out as clearly as they might. Perhaps the most impressive specific instance of the general idea we are discussing here occurs in the work of Krohn-Rhodes [2]. Linear system theory [3,4] itself provides a second example. And a third example can be extracted from the important work of Wei-Norman [5]. The hope is that the synthesis undertaken in an informal way here will make these principles a little more accessible to non-specialists. Moreover while it is perhaps not necessary to treat the examples in as much detail as is done here, the hope is that this too will help lead to a broader understanding of the underlying principles.

In all cases it is the decomposition of the semigroup which reveals the structure of the system. However, we can adopt different rules in

effecting the decomposition and in this way get a very flexible theory meeting a variety of needs. For example, if the mapping semigroup can be given a group structure, then the theory of group decompositions can be invoked to get a decomposition of the dynamics. If the mapping semigroup admits a matrix algebra structure then again theories are available to effect the decomposition.

The class of systems under discussion here are capable of modeling a wide variety of phenomena lying outside the scope of conventional linear systems theory. By way of comparison with linear theory, we might explain our objective as a search for decomposition procedures which parallel the partial fraction expansion method. To emphasize this point we show by example (section 5) how partial fraction expansion decompositions fall out when this procedure is applied to a linear system. We also show how Krohn-Rhodes theory leads to a further decomposition of system structure beyond the partial fraction expansion level.

To many people it has been clear for some time that a broader conception of system theory -- one might say a general system theory -- would be very desirable since technology no longer respects the classical lines of organizing subject material. Characteristic of this trend has been a merging of the continuous with the discrete and a concomitant blurring of the distinction between linear and nonlinear analysis. This paper may be viewed in this context.

A number of algebraic terms are used in the text and examples. Some of these are not common in the control literature and are explained in the appendix. The others can be found in the references cited there.

2. Automata Theory

Many of the ideas which we want to discuss find their clearest and most elementary statement in the setting of finite state systems. In this section we want to recall a few ideas from automata theory which will help to put subsequent developments in perspective.

Suppose we have finite sets U and X together with an evolution equation

$$x(k+1) = \lambda(x(k), u(k)) \quad ; \quad u(k) \in U \quad ; \quad x(k) \in X$$

We call such an object a finite state system. An important concept in the theory of finite state systems is that of the semigroup of the system. This might be explained as follows.

If X has n elements then the total number of maps of X into itself is n^n . Denote this set of maps by $F(X, X)$. Now the subset of $F(X, X)$ consisting of

$$S = \bigcup_{n \geq 0} \bigcup_{u_i \in U} \lambda(\lambda(\lambda \dots \lambda(\lambda(\cdot, u_1), u_2) \dots u_{n-2}), u_{n-1}), u_n) \quad (2.1)$$

can be given a semigroup structure by introducing a multiplication which is just composition of maps. We use \circ to denote multiplication and denote this semigroup by $\mathcal{P} = (S, \circ)$. It is often called the Myhill semigroup. It has only a finite number of elements because $F(X, X)$ is finite.

There is a second semigroup of interest here and that is the free semigroup over U which consists of all finite strings of elements $u_1 u_2 \dots u_p$ with the multiplication operation being concatenation. We denote this semigroup by U^* . Each element in U^* gives rise to exactly

one element of S according to the rule $\lambda^*: u_1 u_2 \dots u_p \rightarrow \lambda(\lambda(\dots \lambda(\cdot, u_1) u_2) \dots u_p)$.

It is immediate that the diagram below is commutative with this definition of λ^* . That is to say, λ^* is a homomorphism of U^* into \mathcal{P}

$$\begin{array}{ccc} U^* \times U^* & \xrightarrow{\text{concatenate}} & U^* \\ \lambda^* \times \lambda^* \downarrow & & \downarrow \lambda^* \\ \mathcal{P} \times \mathcal{P} & \xrightarrow{\text{composition}} & \mathcal{P} \end{array}$$

Since λ^* is onto \mathcal{P} we may say that \mathcal{P} is the homomorphic image of the semigroup U^* .

In semigroups a homomorphism defines a congruence which can be "divided out" to get a simpler semigroup. This point of view gives rise to an alternative characterization of the homomorphism λ^* . If $u_1 u_2 \dots u_q$ is a string which takes all states back to themselves after q steps then the homomorphism λ^* takes this sequence into the identity of \mathcal{P} . Moreover no other strings are taken into the identity of \mathcal{P} so that the kernel of this homomorphism is the set of sequences which give rise to closed paths in the state space for each initial state.[†] In this sense

$$\mathcal{P} = \text{sequences} / (\text{sequences giving closed paths})$$

It is exactly the insertion of the semigroup \mathcal{P} into the theory of finite state systems which makes it possible to study decomposition theory using algebraic methods. In fact the introduction of algebraic

[†]This statement with its topological implications were pointed out by me by Prof. D.L. Elliot of Washington University.

machinery comes about in a very natural way after one more step. Observe that we may associate with each element u_1 of U a map $\lambda(\cdot, u_1)$. If $s(\cdot)$ belongs to \mathcal{P} then the difference equation

$$s(k+1) = [\lambda(\cdot, u(k))] \circ s(k) \quad (2.2)$$

evolves in the semigroup \mathcal{P} . The solution of this equation is "fundamental" in a sense similar to the use of "fundamental solution" in linear theory. That is, if $s(\cdot)$ is the solution corresponding to an initial state which is the identity element of \mathcal{P} and an input string $u_1 u_2 u_3 \dots$, then the solution at time i of the equation

$$x(k+1) = \lambda(x(k), u(k)) \quad ; \quad x(0) = x_0 \quad ; \quad u(\cdot) = u_1 u_2 u_3 \dots$$

is the image of x_0 under the map $s(i)$ viewed as an element of $F(X, X)$.

We call the equation for s the semigroup equation or the Myhill equation. It is important to emphasize that the solution of the semigroup equation evolves in a very simple way, regardless of the complexities of λ . If one knows enough about the structure of finite semigroups the decomposition of this equation into simpler pieces can be carried out. This step has been carried out by Krohn and Rhodes in their important study [2]. In the special case where \mathcal{P} is actually a group the Krohn-Rhodes results on decomposition are not difficult to explain. The idea is that either the group is simple in which case they show that in a certain sense the system is irreducible, or else it is not, in which case the normal subgroups can be divided out to get a decomposed system. We give an example in the next section.

In the remainder of the paper we investigate to what extent we can carry over these ideas to infinite state discrete and continuous time systems.

3. An Example of a Finite Group Decomposition

The examples in this paper progress from the easy to the difficult. Our first example, illustrating the Krohn-Rhodes theory, is interesting because it shows that from the point of view of automata theory a scalar first order difference equation (over a finite field) can sometimes be further decomposed.

Consider the system

$$x(k+1) = \alpha x(k) + \beta u(k) \quad ; \quad y(k) = x(k)$$

where $x(k)$ and $u(k)$ take on the values $0, 1, 2$, and α and β are constants which take on one of these values and arithmetic is done modulo 3.

The total number of maps of the state space into itself is 27 - the semigroup itself consists of a subset of the following (observe that α^3 equals α)

$g_1(\cdot) = \alpha(\cdot)$	$g_7(\cdot) = \alpha^2(\cdot) + \beta$
$g_2(\cdot) = \alpha(\cdot) + \beta$	$g_8(\cdot) = \alpha^2(\cdot) + \alpha\beta + \beta$
$g_3(\cdot) = \alpha(\cdot) + \beta^2$	$g_9(\cdot) = \alpha^2(\cdot) + \alpha\beta^2 + \beta$
$g_4(\cdot) = \alpha^2(\cdot)$	$g_{10}(\cdot) = \alpha^2(\cdot) + \beta^2$
$g_5(\cdot) = \alpha^2(\cdot) + \alpha\beta$	$g_{11}(\cdot) = \alpha^2(\cdot) + \alpha\beta + \beta^2$
$g_6(\cdot) = \alpha^2(\cdot) + \alpha\beta^2$	$g_{12}(\cdot) = \alpha^2(\cdot) + \alpha\beta^2 + \beta^2$

For example, if $\alpha = 2$ and $\beta = 1$ then there are 6 maps which are distinct. Let's take these as g_1, g_2, g_3, g_4, g_5 , and g_6 . A short calculation reveals that this group is isomorphic to the dihedral group* D_3 . We can take g_6 and g_1 to be the generators. Since D_3

*The dihedral group D_n is a group of order $2n$ consisting of all possible products of two generators x and y subject to the relations $x^n=1$, $y^2=1$ and $y x y = x^{-1}$.

is not simple we can decompose this semigroup and the resulting system.

By letting $z(k) = 2^{-k}x(k)$, we can write the evolution equation in terms of modulo 3 arithmetic as

$$z(k+1) = z(k) + \alpha^{-1}w(k)u(k) \quad ; \quad y(k) = w(k)z(k)$$

$$w(k+1) = 2 \cdot w(k)$$

The semigroup of the second of these is isomorphic to *Z_2 whereas the semigroup of the first (regarding $w(k)u(k)$ as the input) is isomorphic to Z_3 . The appropriate block diagrams are shown below.

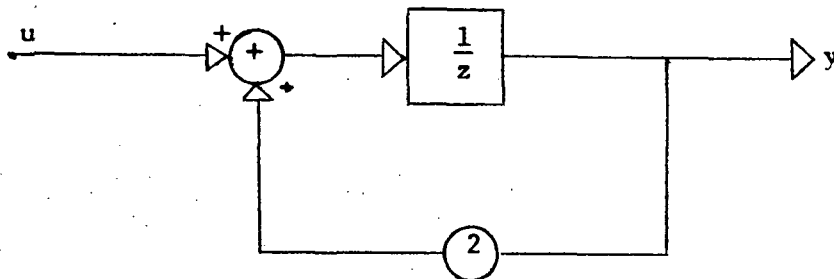


Figure 1 : Linear Sequential Machine Representation of a Modulo 3 System.

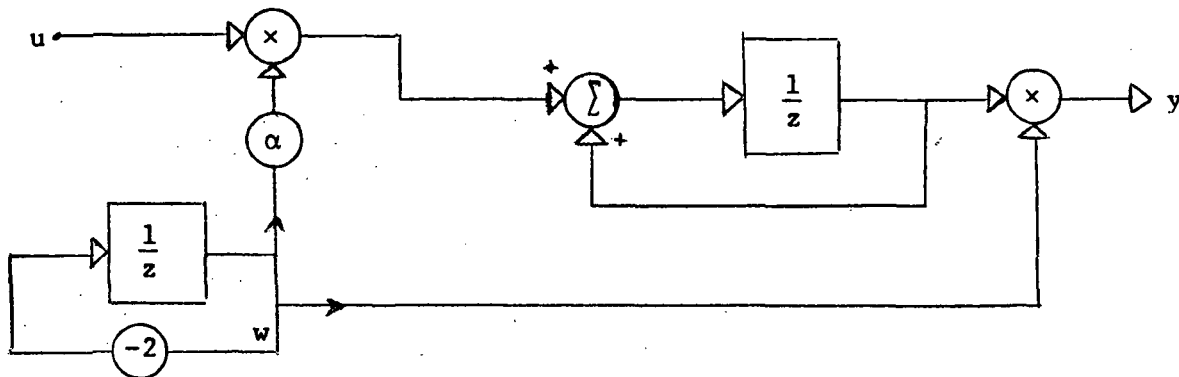


Figure 2 : Decomposed Version of the Modulo 3 System of Figure 1.

*Z_p denotes the group of integers $(0,1,2,\dots,p-1)$ with addition modulo p being the group operation.

4. Bilinear Discrete Time Systems

Even if we abandon the assumptions that U and X be finite sets it is still possible to utilize the previous definitions for \mathcal{S} and the semigroup equation itself. Typically \mathcal{S} will not be finite although there certainly are interesting cases for which it is and in these cases the Krohn-Rhodes theory will apply. The structure of infinite semigroups on the other hand is not well understood and thus to make further progress it is natural to look at systems for which the semigroup admits additional structure. In this section we investigate a class of systems for which it can be given the structure of a matrix algebra.

A significant extension of the linear discrete time system is the class of systems which evolve in a real vector space R^n according to the rule

$$x(k+1) = (A_0 + \sum_{i=1}^v u_i(k)A_i)x(k) + \sum_{i=1}^v b_i u_i(k) \quad (4.1)$$

Here we have a linear dependence on the initial state but a nonlinear dependence on the input. What is the semigroup in this case? Since we have at each step $x(k+1) = M(u)x(k) + n(u)$ it is clear that the set of all maps of the state space into itself is the composition of such maps. However, the composition of two such maps is a third map of the same form. After a calculation one can see that the semigroup for equation (4.1) consists of maps of the form

$$S = \prod_{\ell=0}^{p-1} [A_0 + \sum_{i=1}^v u_i(\ell)A_i]x + \sum_{j=0}^{p-1} \prod_{\ell=j}^{p-1} [A_0 + \sum_{i=1}^v u_i(\ell)] \sum_{i=1}^v b_i u_i(j) \quad (4.2)$$

Recall that a map of \mathbb{R}^n into \mathbb{R}^n is called affine if it is of the form of a translation plus a nonsingular linear transformation. This set of maps would be affine if the linear transformation part were invertible. There

is, however, no need to require invertibility at this point. We call maps of the form $Mx+b$ with M not necessarily invertible, pseudo-affine. Notice that the semigroup defines an equivalence relation on the input space whereby $u_1 \sim u_2$ if they both give rise to the same map.

It is easy to see that it is possible to put the set of pseudo-affine maps in one to one correspondence with a set of $n+1$ by $n+1$ matrices according to the rule

$$\begin{bmatrix} G & b \\ 0 & 1 \end{bmatrix} \sim g \text{ with } g(x) = Gx + b$$

The set of pseudo-affine maps on \mathbb{R}^n is, of course, a semigroup under composition. The correspondence defined above is a semigroup homomorphism if we regard the set of matrices as a multiplicative semigroup. This hinges on the two calculations which give the effect of semigroup multiplication in the respective cases

$$i) \quad \begin{bmatrix} G_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G_2 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} G_1 G_2 & G_1 b_2 + b_1 \\ 0 & 1 \end{bmatrix}$$

$$ii) \quad G_1[G_2 + b_2] + b_1 = G_1 G_2 + G_1 b_2 + b_1$$

We denote the matrix semigroup by

$$\mathcal{A}(n) = \left\{ G: G = \begin{bmatrix} S & x \\ 0 & 1 \end{bmatrix} \right\}$$

Having a convenient representation for the semigroup associated with equation (4.1), the next step is to display the semigroup equation itself. A little thought will verify that the semigroup (4.2) evolves according to the equation

$$S(k+1) = \begin{pmatrix} A_0 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^v u_i(k) \begin{pmatrix} A_i & b_i \\ 0 & 0 \end{pmatrix} S(k) \quad (4.3)$$

By a matrix algebra (over a fixed field) we mean a set of square matrices which is a vector space with respect to matrix addition and scalar multiplication and which is closed under matrix multiplication. Since a lot is known about the structure of matrix algebras including the extent to which they can be decomposed, the question naturally arises as to whether or not these results can be brought to bear. Clearly the semigroup is closed under multiplication; after all this is the semigroup property. Troubles arise with regard to the vector space structure. Even in the special case where the evolution equation is

$$x(k+1) = \left[\sum_{i=1}^v u_i(k) A_i \right] x(k)$$

and the semigroup equation is

$$S(k+1) = \left[\sum_{i=1}^v u_i(k) A_i \right] S(k)$$

the semigroup is in general not closed under matrix addition.

Confronted with this situation a natural thing to do is compute the semigroup and find the smallest matrix algebra which contains it. In fact this seemingly ad hoc solution can be justified further by noticing that if we want to obtain bilinear subsystems this is an appropriate structure. In a complete theory this point will require careful attention. Decomposing this algebra will, of course, decompose the actual semigroup although this procedure overlooks the possibility that the semigroup might admit a decomposition not shared by the smallest matrix algebra which contains it.

What then is the smallest matrix algebra \mathcal{M} containing the set of matrices

$$\mathcal{P} = \bigcup_{u_1 \in \mathbb{R}} \bigcup_{n \geq 0} \prod_{i=0}^n \begin{bmatrix} A_0 + \sum u_1(k) A_1 & \sum u_1(k) b_1 \\ 0 & 1 \end{bmatrix}$$

One can't be more explicit than to display it as

$$\mathcal{M} = \{M : M = \sum \alpha_i S_i, \alpha_i \in \mathbb{R}, S_i \in \mathcal{P}\}$$

except in special cases. For example if A_0 is n by n and if we have

$$\mathcal{P} = \bigcup_{u \in \mathbb{R}} \bigcup_{n \geq 0} \prod_{i=0}^n \begin{bmatrix} A_0 u(k) b \\ 0 & 1 \end{bmatrix}$$

then

$$\mathcal{M} = \{M : M = \begin{bmatrix} \alpha(A_0) & x \\ 0 & \alpha(1) \end{bmatrix} ; \alpha = \text{polynomial of degree } n, x \in \text{Range } b, A_0 b \dots A_0^{n-1} b\}$$

as is easily verified by use of the Cayley-Hamilton theorem.

By bringing standard algebraic decomposition theorems to bear on this problem we can decompose the semigroup and hence obtain a realization of the original system which is decomposed. To make this important point clear, suppose that we can decompose the enlarged semigroup \mathcal{M} as a direct sum of say n parts, $M_1 \oplus M_2 \oplus \dots \oplus M_n$. Then we can write the semigroup equation as

$$\begin{aligned} M_1(k+1) &= [A_0^1 + \sum u_1(k) A_1^1] M_1(k) \\ M_2(k+1) &= [A_0^2 + \sum u_1(k) A_1^2] M_2(k) \\ &\dots \dots \dots \\ M_n(k+1) &= [A_0^n + \sum u_1(k) A_1^n] M_n(k) \end{aligned} \quad (\text{superscripts are not powers})$$

with $s(k) = \Sigma M_1(k)$. Since $x(k) = s(k)x_0$ this set of systems obviously simulates the original system but is decomposed in the sense of having semigroups which are subsets of simple matrix algebras.

5. An Example of a Matrix Algebra Decomposition

Our objective here is to show what this philosophy yields when we apply it to a standard situation.

Consider a linear system

$$x(k+1) = Ax(k) + bu(k) \quad ; \quad x(k) \in \mathbb{R}^n \quad ; \quad u \in \mathbb{R}^1$$

As we have seen the Myhill equation can be expressed as

$$S(k+1) = \begin{bmatrix} A & u(k)b \\ 0 & 1 \end{bmatrix} S(k)$$

The set of matrices

$$\mathcal{S} = \bigcup_{n \geq 0} \prod_{k=0}^n \begin{bmatrix} A & u(k)b \\ 0 & 1 \end{bmatrix}$$

do not form a matrix algebra since it is not closed under addition.

However if we enlarge it by inserting a $v(k)$ to get

$$\mathcal{R} = \bigcup_{n \geq 0} \prod_{k=0}^n \begin{bmatrix} v(k)A & u(k)b \\ 0 & v(k) \end{bmatrix}$$

Then we do get a matrix algebra. More concretely, \mathcal{R} consists of matrices of the form

$$\begin{bmatrix} p(A) & x \\ 0 & p(1) \end{bmatrix}$$

where p is any polynomial of degree n or less and x is any vector in the range space of $b, ab, \dots, A^v b$ with v the degree of p .

We can decompose this matrix algebra to get a decomposition of the original system. This works in the following way. Notice that if A has a diagonal

Jordan normal form then by the transformation

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

we can bring A into diagonal form. Thus we have a matrix algebra whose elements are of the form

$$\begin{bmatrix} p(\lambda_1) & 0 & \dots & 0 & x_1 \\ 0 & p(\lambda_2) & \dots & 0 & x_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p(\lambda_n) & x_n \\ 0 & 0 & \dots & 0 & p(1) \end{bmatrix}$$

where $x = (x_1, x_2, \dots, x_n)'$ is a vector in the reachable set for the transformed system. Since the matrices of the form

$$R_k(p, x) = \begin{bmatrix} \vdots & \vdots \\ \dots & p(\lambda_k) \dots \dots x_k \\ \vdots & \vdots \\ \dots & 0 \dots \dots p(1) \end{bmatrix}$$

form a one sided ideal, $(R_k(p, x) \cdot R_l(p, x) \subset R_k(p, x))$, it is easily verified that

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \dots + \mathcal{R}_n$$

where $+$ indicates a semidirect decomposition in the sense of matrix algebras. We leave the details of the repeated root case to the reader.

* That is to say the \mathcal{R}_i are ideals which as vector spaces taken all together span \mathcal{R} . However the vector spaces \mathcal{R}_i are not necessarily orthogonal as they would be in a direct sum decomposition.

6. Bilinear Continuous Time Systems

Carrying these ideas over to the case of ordinary differential equations is not as difficult as one might suppose. The assumptions we use to insure that the semigroup will have a manageable form are very similar to those used in section 4. Instead of matrix algebras, matrix Lie algebras are the key to understanding the structure.

We consider systems of the form

$$\dot{x}(t) = [A_0 + \sum_{i=1}^v u_i(t)A_i]x(t) + \sum_{i=1}^v b_i u_i(t) \quad (6.1)$$

Notice that the input-output maps of such systems are decidedly nonlinear and this class is not as special as it might look at first sight. Moreover, this class of models fill an important gap in the currently available theory because they allow one to model systems for which the Euclidean norm $||x|| = (\sum x_i^2)^{1/2}$ is preserved and also allow one to model systems for which the ℓ_1 norm $||x|| = \sum |x_i|$ is preserved. The former condition has significant application in systems where energy is conserved and the latter is important in modeling continuous time jump processes where the sum of the probabilities is necessarily one. Systems in which either constraint is an important aspect obviously cannot be modeled as

$$\dot{x}(t) = Ax + bu(t)$$

with the system being controllable. Rink and Mohler [6] and the author [7] cite further applications of this model.

A good deal is known about the controllability of equation 6.1 as the result of Lie algebraic techniques, [7-10]. It follows from the variation

of constants formula that the set of maps of the state space into itself are all of the form $x \mapsto Mx+b$. The exact set of M 's which can appear here are the set of possible transition matrices and the set of b 's depend on the reachable set. Of course if we augment x by adding an additional component which is always one, then we have

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ 1 \end{bmatrix} = \begin{bmatrix} A_0 + \sum u_i(t) A_i & \sum u_i(t) b_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ 1 \end{bmatrix}$$

This device allows us to think of 6.1 as being a special case of

$$\dot{x}(t) = [A_0 + \sum_{i=1}^v u_i(t) A_i] x(t) \quad (6.2)$$

It is clear that the analog of the Myhill equation appropriate for equation (6.2) is the matrix equation

$$\dot{S}(t) = [A_0 + \sum_{i=1}^v u_i(t) A_i] S(t) \quad (6.3)$$

The possibilities for decomposing this equation are implicit in the very interesting work of Wei and Norman [5] on the solution of time varying linear differential equations. What Wei and Norman show is that the smallest vector space of matrices which is closed under the operation of commutation, $[A,B] = AB-BA$, plays a decisive role. This space is called a Lie algebra and it plays an important role here and in related work [7-10].

The relationship between the commutator and structure of the solution of linear differential equations may be explained as follows. First of all it is known (see e.g. Wichmann [11]) that if for each i , A_i is a piecewise continuous function of time for $-\infty < t < \infty$ and if

$$\dot{x}(t) = \left[\sum_{i=1}^r A_i(t) \right] x(t)$$

then the transition matrices Φ_{A_i} of $\dot{x}(t) = A_i(t)x(t)$; are related to the transition matrix of the total system via

$$\Phi_{\Sigma A_i} = \Phi_{A_1} \Phi_{A_2} \dots \Phi_{A_r}$$

with the individual factors on the right commuting, provided that for all i and j $[A_i, A_j] = 0$.

The proof of this is easy in the case $v = 2$ and the general result follows by an induction.

Secondly, it is known (see Wichmann [11] or Wei-Norman [5]) that if the Lie algebra generated by a set of constant matrices $\{A_i\}$ is solvable* then the solution of the differential equation $\dot{x}(t) = [g_1(t)A_1 + \dots g_v(t)A_v]x(t)$ can be expressed explicitly in terms of integrals.

The preceding remarks lead to the conclusion that the basic solution structure stands revealed in the decomposed version of the Lie algebra generated by the A_i . If this is a semi-simple algebra then

$$\mathcal{P} = (\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3, \dots \oplus \mathcal{P}_r)$$

where the \mathcal{P}_i are simple subalgebras, and the previous analysis shows that the transition matrix is

$$\Phi = X_r X_{r-1} \dots X_1$$

where the factors X_i belong to the Lie groups corresponding to the simple Lie algebras \mathcal{P}_i . If the algebra has a radical in addition

* See appendix for a definition.

to the semisimple part then provided that one can compute the solution for the simple subalgebras one can arrive at an equation involving the radical which can be solved explicitly. In order to actually solve the equation when the subalgebras are not solvable Wei and Norman suggest looking for a solution of the form

$$X(t) = e^{g_1(t)H_1} e^{g_2(t)H_2} \dots e^{g_r H_r}$$

What their method rests on is the demonstration of the following fact.

Let H_1, \dots, H_n be a basis for \mathcal{L} . Then

$$\prod_{j=1}^r e^{g_j H_j} H_1 \prod_{j=r}^1 e^{-g_j H_j} = \sum_{k=1}^n \xi_{k1} H_k ; \quad r=1, \dots, n$$

where each of the ξ_{k1} is an analytic function of g_1, g_2, \dots, g_r . Having this at their disposal it is easy to verify that at least for small $|t|$ one can find a solution in the given form simply by equating the coefficients of L_1 on each side of the equation

$$\frac{d}{dt} e^{g_1 H_1} e^{g_2 H_2} \dots e^{g_r H_r} = A e^{g_1 H_1} e^{g_2 H_2} \dots e^{g_r H_r}$$

7. An Example of a Lie Group Decomposition

Consider the electrical network shown in figure 1. This model illustrates some of the features of a voltage conversion network. The equations of motion are ($u = 0$ corresponds to left switch open and right switch closed, $u=1$ corresponds to left switch closed, right switch open)

$$\begin{aligned} C_1 \dot{V}_1 &= u I_3 \\ C_2 \dot{V}_2 &= -(1-u) I_3 + I \\ L \dot{I}_3 &= u(E - V_1) + (1-u) V_2 \end{aligned}$$

Now if we make the replacements $x_1 = \sqrt{C_1} V_1$; $x_2 = \sqrt{C_2} V_2$ and $x_3 = \sqrt{L} I_3$ and let $\alpha = 1/\sqrt{LC_2}$, $\beta = 1/\sqrt{LC_1}$, $\gamma = E/\sqrt{L}$, $\delta = I/\sqrt{C_2}$

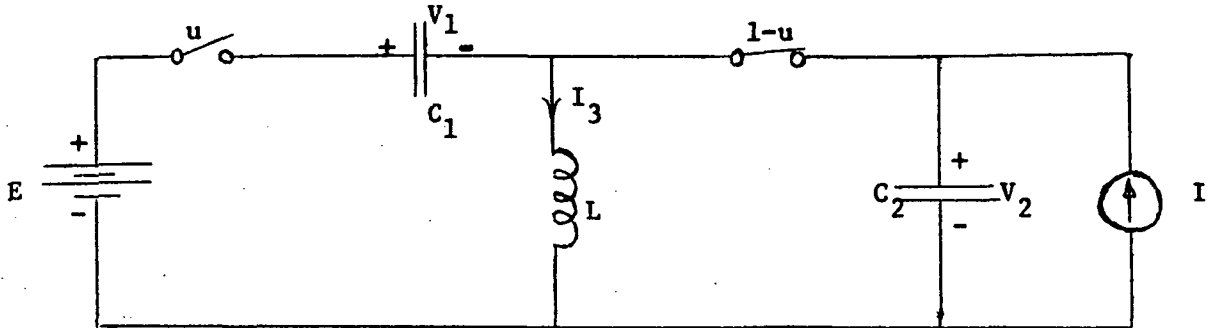


Figure 3 : An electrical network controlled by switches

then we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & +\alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ -\beta & -\alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta \\ u\gamma \end{bmatrix}$$

We now introduce the affine representation and write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & \delta \\ 0 & +\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & \alpha & 0 \\ -\beta & -\alpha & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The smallest Lie algebra which contains these two matrices is a 6 dimensional algebra whose typical element is

$$\begin{bmatrix} 0 & \omega_3 & \omega_2 & \mu \\ -\omega_3 & 0 & \omega_1 & \nu \\ -\omega_2 & -\omega_1 & 0 & \rho \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This Lie algebra contains as a three dimensional ideal the subalgebra whose typical element is

$$\begin{bmatrix} 0 & \omega_3 & \omega_2 & 0 \\ -\omega_3 & 0 & \omega_1 & 0 \\ -\omega_2 & -\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we can decompose the Lie algebra as

$$\mathcal{L} = \mathcal{K}_1 + \mathcal{R}_1 + \mathcal{R}_1 + \mathcal{R}_1$$

where \mathcal{R}_1 indicates the one dimensional Lie algebra and + indicates a semidirect product. Let S_1 be the solution of the equation

$$\dot{S}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{bmatrix} S_1 + u \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ -\beta & -\alpha & 0 \end{bmatrix} S_1 ; S_1(0) = I$$

and let $b_1 = (0, \delta, 0)'$ and $b_2 = (0, 0, \gamma)'$. Then the block diagram of the decomposed system is shown in figure 4.

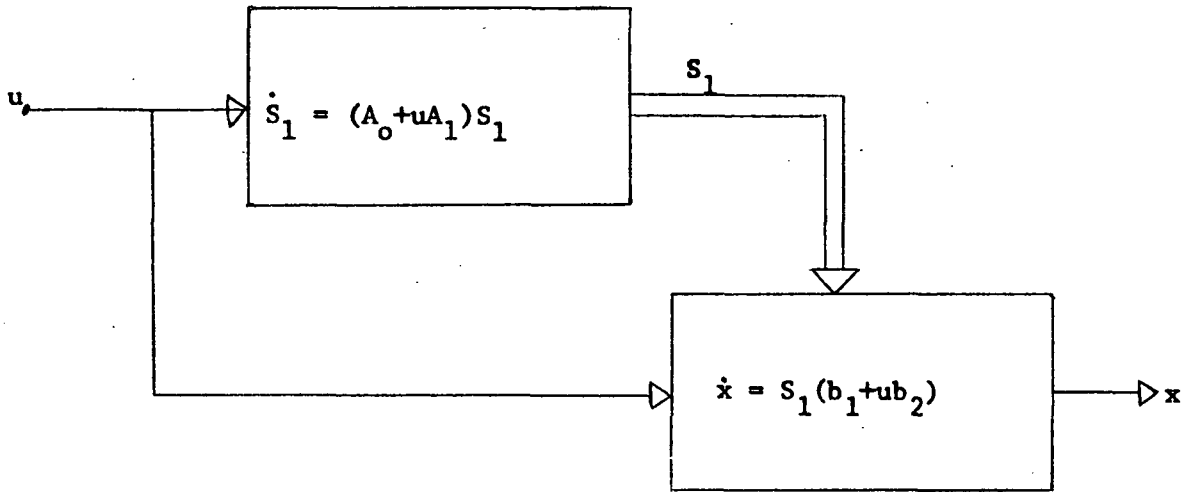


Figure 4 : Showing the decomposed version of systems in block diagram form

Perhaps it is of some interest to carry this analysis a little bit further to give a more complete picture of the Wei-Norman method. To do this we pick a basis for \mathcal{K} and proceed as follows.

Let Ω_1 , Ω_2 and Ω_3 be given by

$$\Omega_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{bmatrix}; \quad \Omega_2 = \begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \quad \Omega_3 = \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly these generate a Lie algebra which is not solvable. We note that a direct power series expansion together with the identities

$$\Omega_1 = [\Omega_2, \Omega_3], \quad \Omega_2 = [\Omega_3, \Omega_1], \quad \Omega_3 = [\Omega_1, \Omega_2], \quad \text{gives}$$

$$\begin{aligned} e^{\alpha\Omega_1} \Omega_2 e^{-\alpha\Omega_1} &= \Omega_2 + \alpha[\Omega_1, \Omega_2] + \frac{1}{2!} \alpha^2 [\Omega_1, [\Omega_1, \Omega_2]] + \dots \\ &= \Omega_2 + \alpha\Omega_3 - \frac{1}{2!} \alpha^2 \Omega_2 - \frac{1}{3!} \alpha^3 \Omega_3 + \dots \\ &= \Omega_2 \cos \alpha + \Omega_3 \sin \alpha \end{aligned}$$

and also

$$\begin{aligned} e^{\alpha\Omega_2} \Omega_3 e^{-\alpha\Omega_2} &= \Omega_3 \cos \alpha + \Omega_1 \sin \alpha \\ e^{\alpha\Omega_3} \Omega_1 e^{-\alpha\Omega_3} &= \Omega_1 \cos \alpha + \Omega_2 \sin \alpha \end{aligned}$$

On the other hand,

$$\begin{aligned} e^{\alpha\Omega_3} \Omega_2 e^{-\alpha\Omega_3} &= \Omega_2 \cos \alpha - \Omega_1 \sin \alpha \\ e^{\alpha\Omega_1} \Omega_3 e^{-\alpha\Omega_1} &= \Omega_3 \cos \alpha - \Omega_2 \sin \alpha \\ e^{\alpha\Omega_2} \Omega_1 e^{-\alpha\Omega_2} &= \Omega_1 \cos \alpha - \Omega_3 \sin \alpha \end{aligned}$$

Now if we try for a solution of

$$\dot{X} = (u_1(t)\Omega_1 + u_2(t)\Omega_2)X ; X(0) = I$$

in the Wei-Norman form we assume $X = e^{g_1\Omega_1} e^{g_2\Omega_2} e^{g_3\Omega_3}$ we have

$$\dot{X} = \dot{g}_1\Omega_1 e^{g_1\Omega_1} e^{g_2\Omega_2} e^{g_3\Omega_3} + e^{g_1\Omega_1} \dot{g}_2\Omega_2 e^{g_2\Omega_2} e^{g_3\Omega_3} + e^{g_1\Omega_1} e^{g_2\Omega_2} \dot{g}_3\Omega_3 e^{g_3\Omega_3}$$

Now use the above to get

$$\begin{aligned} e^{g_1\Omega_1} \dot{g}_2\Omega_2 e^{g_2\Omega_2} e^{g_3\Omega_3} &= \dot{g}_2 e^{g_1\Omega_1} \Omega_2 e^{-g_1\Omega_1} \\ &= \dot{g}_2 (\Omega_2 \cos g_1 + \Omega_3 \sin g_1) X \end{aligned}$$

Use this idea twice to get

$$\begin{aligned} \dot{g}_3 e^{g_1\Omega_1} e^{g_2\Omega_2} \Omega_3 e^{g_3\Omega_3} &= \dot{g}_3 e^{g_1\Omega_1} (\Omega_3 \cos g_2 + \Omega_1 \sin g_2) e^{g_2\Omega_2} e^{g_3\Omega_3} \\ &= \dot{g}_3 [\Omega_1 \sin g_2 + \cos g_2 (\Omega_3 \cos g_1 - \Omega_2 \sin g_1)] X \end{aligned}$$

so the Wei-Norman Equations are in matrix form

$$\begin{aligned} \dot{g}_1\Omega_1 + \dot{g}_2(\Omega_2 \cos g_1 + \Omega_3 \sin g_1) + \\ \dot{g}_3(\Omega_1 \sin g_2 + \Omega_3 \cos g_1 \cos g_2 - \Omega_2 \cos g_2 \sin g_1) &= u_1\Omega_1 + u_2\Omega_2 \end{aligned}$$

Decomposed these become

$$\dot{g}_1 + \dot{g}_3 \sin g_2 = u_1$$

$$\dot{g}_2 \cos g_1 - \dot{g}_3 \cos g_2 \sin g_1 = u_2$$

$$\dot{g}_2 \sin g_1 + \dot{g}_3 \cos g_1 \cos g_2 = 0$$

and finally

$$\begin{bmatrix} 1 & 0 & \sin g_2 \\ 0 & \cos g_1 & (-\cos g_2 \sin g_1) \\ 0 & \sin g_1 & (\cos g_1 \cos g_2) \end{bmatrix} \begin{bmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

Notice that

$$\det \begin{bmatrix} \cos g_1 & -\cos g_2 \sin g_1 \\ \sin g_1 & \cos g_1 \cos g_2 \end{bmatrix} = \cos^2 g_1 \cos g_2 + \sin^2 g_1 \cos g_2 = \cos g_2$$

This set of equations therefore is not meaningful at $g_2 = \pm\pi/2$.

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9. Appendix on Algebraic Structures

The purpose of this appendix is to collect a few facts about groups, associative algebras and Lie algebras so as to make it easier for the reader to make contact with the literature. All the definitions needed for sections 2 and 3 are contained in Chapter 7 of reference [3]. Otherwise the book by Rotman [12] is very readable. For algebras (sections 4 and 5) see for example Greub [16] and Gray [13] and for Lie algebras (sections 6 and 7) Samelson [14] and Jacobson [15] are appropriate.

A groupoid is a pair (S, \cdot) where S is a set and \cdot is a binary operation $\cdot : S \times S \rightarrow S$. If this binary operation is associative i.e. if $(s_1 \cdot s_2) \cdot s_3 = s_1 \cdot (s_2 \cdot s_3)$, then (S, \cdot) is a semigroup. A monoid is a semigroup in which there exists an element e such that for all s in S , $es = se = s$. Monoids which have the additional property that for each s in S there exist t in S such that $st = ts = e$ are called groups. An abelian group is a group such that $s \cdot t = t \cdot s$ for all s and t in S . A group (R, \cdot) is said to be a subgroup of (S, \cdot) if R is a subset of S and the multiplication is the same on R as in S . The order of a group is the number of elements in it.

If (S, \cdot) and $(R, *)$ are semigroups and h is a mapping $h : S \rightarrow R$ we say that h is a homomorphism if the diagram below "commutes" i.e. is consistent.

$$\begin{array}{ccc} S \times S & \xrightarrow{\cdot} & S \\ \downarrow h \times h & & \downarrow h \\ R \times R & \xrightarrow{*} & R \end{array} \quad s_1 s_2 = s_3 \Rightarrow h(s_1)h(s_2) = h(s_3)$$

A homomorphism which is one to one (as opposed to many to one) and onto is called an isomorphism.

Now let S be a group and R a subgroup. That is, suppose that there is an insertion i such that

$$R \xrightarrow{i} S$$

is one to one. We can see that the statement $s_1 \sim s_2$ if and only if there exists r in R such that $s_1 r = s_2$, defines an equivalence relation on S and hence a partition on S . We call the elements of this partition cosets. A subgroup R of S is said to be a normal subgroup if $r \in R$ and $s \in S$ means $srs^{-1} \in R$ which is to say; $sR = Rs$ for each s in S . We say that a group is simple if its only normal subgroups are itself and the trivial group consisting of the identity. We will not discuss the decomposition theorems available for groups since this is done in the present context elsewhere [3].

An algebra \mathcal{P} is a triple $(S, +, \cdot)$ where $(S, +)$ is a vector space over a field and \cdot is a bilinear multiplication. If $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ for all A, B and C in S then the algebra is said to be associative. Perhaps the most common example of an associative algebra is the algebra of n by n matrices with $+$ and \cdot being matrix addition and matrix multiplication. A Lie algebra (discussed below) is an example of a nonassociative algebra. By a subalgebra of \mathcal{P} we mean an algebra $\mathcal{P}_1 \subset \mathcal{P}$ such that $\mathcal{P}_1 \cdot \mathcal{P}_1 \subset \mathcal{P}_1$ and $\mathcal{P}_1 + \mathcal{P}_1 \subset \mathcal{P}_1$. A subalgebra is called an ideal if $\mathcal{P}_1 \cdot \mathcal{P} \subset \mathcal{P}_1$. Clearly the sum of two ideals is an ideal. An ideal \mathcal{P}_1 is called nilpotent if for each s in

\mathcal{P}_1 there is an n such that $s^n = 0$. The sum of all the nilpotent ideals is called the radical. By a matrix algebra we mean a set of matrices which is closed under addition and multiplication which forms a vector space over its field of definition.

A Lie algebra is an algebra in which $(S, +)$ is a vector space and in which the product (denoted by $[,]$) is bilinear, that is, for x, y and z in S we have $[(x+y), z] = [x, z] + [y, z]$: $[x, (y+z)] = [x, y] + [x, z]$ and $\alpha[x, y] = [\alpha x, y] = [x, \alpha y]$. In addition $[,]$ is required to satisfy the conditions $[x, x] = 0$, $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. The latter condition, known as the Jacobi identity, is the substitute for associativity.

We need only be concerned with Lie algebras for which S is a set of n by n matrices whose entries are real numbers. The Lie product is the commutator $[X, Y] = XY - YX$. It is easy to see that this product satisfies the above conditions.

Let $\{H_1\}$ be a set of n by n matrices; the Lie algebra generated by $\{H_1\}$ consists of $\{H_1\}$, all the elements obtained from $\{H_1\}$ by repeated commutations, and all the linear combinations of these. A subalgebra \mathcal{P} of a given algebra is called an ideal if $[\mathcal{P}, \mathcal{L}] \subset \mathcal{P}$ i.e., for all $X \in \mathcal{P}$ and $Y \in \mathcal{L}$ the product $[X, Y]$ belongs to \mathcal{P} .

The set of all elements of \mathcal{L} which are the result of commutation of some two elements form the derived algebra. This is denoted by \mathcal{L}' . Clearly \mathcal{L}' is an ideal of \mathcal{L} . The derived algebra of \mathcal{L}' is denoted by \mathcal{L}'' . Continuing, we have the derived series

$$\mathcal{L}' \subset \mathcal{L}'' \dots \subset \mathcal{L}^{(h)} \subset \mathcal{L}^{(h+1)} \dots$$

A Lie algebra \mathcal{L} is said to be solvable if $\mathcal{L}^{(h)} = \{0\}$ for some h .

The sum of two solvable ideals is again a solvable ideal. The radical of \mathcal{L} is the sum of all of its solvable ideals.

The Lie algebra \mathcal{L} is said to be semisimple if its radical is $\{0\}$. It is called simple if it has no ideal other than \mathcal{L} and $\{0\}$, and if $\mathcal{L}' \neq 0$. The last condition serves to avoid trivial cases.

The main source of knowledge about the structure of associative algebras comes from Wedderburn's theorem. This result can be found in reference [13] as a statement about rings.

There are two main structure theorems of Lie algebras. The first, known as Levi's Theorem states that if \mathcal{L} is a finite dimensional Lie algebra with radical \mathcal{L}_0 , then there exists a semisimple subalgebra $\mathcal{L}_1 \subset \mathcal{L}$ such that given $X \in \mathcal{L}$, there exist unique $X_0 \in \mathcal{L}_0$, and unique $X_1 \in \mathcal{L}_1$ such that $X = X_0 + X_1$. For the proof of this theorem see Jacobson, [15]. The second structure theorem explains what happens to the semisimple part and goes like this. A finite dimensional semisimple Lie algebra \mathcal{L} may be decomposed into the direct sum $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_r$, where the \mathcal{L}_i are ideals which are simple algebras.

10. Appendix on Linear Continuous Time Systems

Consider the standard time invariant linear system $x(t) \in \mathbb{R}^n$,
 $u(t) \in \mathbb{R}^m$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ; \quad y(t) = Cx(t) \quad (10.1)$$

Suppose that we assume that this system is controllable and observable.

Now consider the set of all possible maps of the state at $t = 0$ into the state at sometime later which u can generate. Clearly these maps are of the form

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma$$

which is an affine map. This set of maps, which constitute the semigroup of the system, satisfy a very simple differential equation of the form $\dot{S}(t) = U(t)S(t)$. More specifically,

$$\frac{d}{dt} \begin{bmatrix} e^{At} & x_a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & Bu(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{At} & x_a \\ 0 & 1 \end{bmatrix}$$

where

$$x_a(t) = \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma$$

The subset of the n -dimensional affine group which consists of

$$\mathcal{S} = \bigcup_{t \geq 0} \begin{bmatrix} e^{At} & x \\ 0 & 1 \end{bmatrix} \quad ; \quad x \in \text{Range of } \{B, AB, \dots, A^{n-1}B\}$$

is in general not a group since t is restricted to be nonnegative. It will be called the semigroup of the linear system by analogy with the standard definition of the semigroup of a machine in automata theory.

Notice that having the solution of the semigroup equation

$$\frac{d}{dt} \begin{bmatrix} s_{11}(t) & s_{12}(t) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & Bu(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_{11}(t) & s_{12}(t) \\ 0 & 1 \end{bmatrix}$$

with the initial condition being the identity matrix (the identity in $\mathcal{A}(n)$)

gives the solution of equation (10.1) via the rule

$$\begin{bmatrix} x(t) \\ 1 \end{bmatrix} = S(t) \begin{bmatrix} x(0) \\ 1 \end{bmatrix}$$

AVERAGE VALUE CRITERIA FOR STOCHASTIC STABILITY

Roger W. Brockett*
Division of Engineering
and Applied Physics
Harvard University
Cambridge, Mass. 02138, U.S.A.

Jan C. Willems**
Department of Electrical Engineering
Massachusetts Institute of Technology
Cambridge, Mass. 02139, U.S.A.

INTRODUCTION

Many problems in control and other areas of applied mathematics lead to stability questions for dynamical systems which are described by mathematical models involving time-varying parameters. Frequently one may assume that these time-varying parameters are stochastic processes with known statistics. Typical examples of interesting applications which lead to such stochastic stability questions are the stability analysis of numerical computations in the face of round-off error, systems involving the human operator, sampled data systems with jitter in the sampling rate, mechanical systems subject to random vibrations, and economic systems which model some of the uncertainties as variable lags.

Essentially all of the above examples lead to mathematical models in which the stochastic processes enter the model in a multiplicative way. It is for this class of systems that the stochastic stability question becomes interesting and challenging. In contrast, when the stochastic processes enter the model in an additive way as, for example, in the linear quadratic theory, then the stochastic stability question usually reduces to the stability of the deterministic system obtained by putting the stochastic processes equal to zero.

In this paper we will analyze a class of stochastic systems and obtain various explicit stability criteria. Before we describe the model let us introduce the following notation: R denotes the real number system, R^n denotes n -dimensional real Euclidean space, $R^{m \times p}$ denotes the real $m \times p$ matrices, prime denotes transpose,

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≥ 0 (> 0) means that a symmetric matrix is nonnegative (positive) definite, $\lambda[\cdot]$ denotes an arbitrary eigenvalue of a matrix, whereas $\lambda_{\max}[\cdot]$ ($\lambda_{\min}[\cdot]$) denotes the maximum (minimum) eigenvalue of a matrix with real eigenvalues, Re denotes the real part of a complex number, $\max[\cdot, \cdot]$ ($\min[\cdot, \cdot]$) denotes the maximum (minimum) of two real numbers, and $\mathcal{E}\{\cdot\}$ denotes the expected value of a random variable.

We will study the stability of the linear system Σ described by the differential equation:

$$\Sigma : \dot{x} = Ax - BK(t)Cx,$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ are constant matrices and $K(t)$ is a time-varying function taking values in $\mathbb{R}^{m \times p}$. The differential equation Σ will be viewed as describing the closed loop dynamics of the feedback interconnection of the stationary linear system

$$\Sigma_1 : \dot{x}_1 = Ax_1 + Bu_1 ; y_1 = Cx_1$$

in the forward loop, and the memoryless time-varying linear system

$$\Sigma_2 : y_2 = K(t)u_2$$

in the feedback loop. The feedback interconnection equations are given by:

$u_1 = -y_2$, $u_2 = y_1$. It is easily verified that we indeed have $\Sigma = \Sigma_1 \Sigma_2$ feedback.

This feedback system is shown in Figure 1.

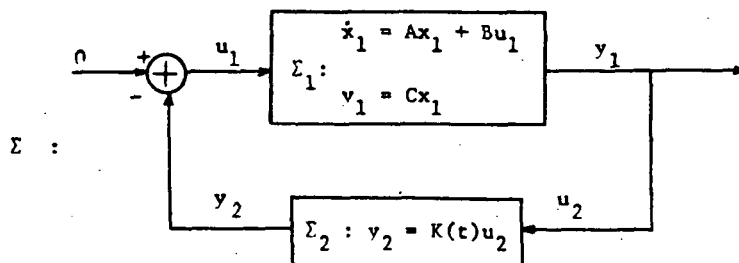


Figure 1: Σ viewed as $\Sigma_1 \Sigma_2$ feedback.

We will assume throughout, for simplicity, that $\Sigma_1 = \{A, B, C\}$ is minimal (i.e., (A, B) is controllable and (A, C) is observable). The transfer function of Σ_1 is given by $G(s) = C(Is - A)^{-1}B$. The gain matrix $K(t)$ is assumed to be a stochastic process whose properties will be described in more detail later. We seek conditions

on the statistics of $K(t)$ which guarantee the stability of Σ (to be defined later).

If we consider the equation for Σ from a state space point of view then it is apparent that the case where $K(t)$ is a colored process is quite distinct from the case that $K(t)$ is white. If $K(t)$ is white noise then the system behaves pretty much like a linear one and we may use most of the theory on stochastic differential equations directly as for example the Lyapunov techniques for stochastic systems (see e.g., Kushner [1967], Chapter 2). If on the other hand $K(t)$ is a colored process then we should model Σ as something like:

$$\dot{z} = Fz + Gw ; \quad K = Hz$$

$$\dot{x} = Ax - BKCx$$

with w white noise. This case is thus inherently nonlinear. The results obtained in this paper fall into two categories. In the first class we consider the colored case and show how one may use what are essentially linear techniques to obtain conditions for almost sure asymptotic stability of Σ . The method of proof uses Wazewski's inequality previously exploited in this context by Infante [1968]. These criteria are thus independent of the autocorrelation function of $K(t)$.

The second class of results considers the white noise case and shows how one may use the frequency-domain stability criteria for linear systems in order to obtain criteria for mean square stability of Σ . This question has been studied extensively in the literature and the results obtained here complement those obtained by Willems and Blankenship [1971] and Willems [1972].

1. AVERAGE VALUE CRITERIA FOR ALMOST SURE STOCHASTIC STABILITY

In this section we will assume that the entries of the gain matrix $K(t) \in \mathbb{R}^{m \times p}$ are stationary stochastic processes satisfying an ergodicity hypothesis which ensures the almost sure equality of time averages and ensemble averages. Thus if $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is integrable then we assume that almost surely:

$$\mathcal{E}\{F(K(t))\} = \mathcal{E}\{F(K(0))\} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} F(K(\tau)) d\tau .$$

We will consider almost sure asymptotic stability. This is defined as:

Definition 1: Σ is said to be *almost surely asymptotically stable* if the equality

$\lim_{t \rightarrow \infty} x(t) = 0$ holds with probability one for all given initial conditions $x(t_0)$.

1.1 A Stability Criterion for Completely Symmetric Systems

Consider the system Σ_1 . There are various ways of describing its response function from the inputs to the outputs. The most commonly used input/output descriptions of Σ give either its transfer function $G(s) \triangleq C(Is-A)^{-1}B$ or its impulse response $W(t) \triangleq Ce^{At}B$ ($t \geq 0$). There is however an alternative input/output description which, although it has roots going back at least as far in time as do the concepts of transfer function and impulse response, has become particularly prevalent in the last half decade. This description gives the so-called *Hankel matrix* of Σ_1 defined by:

$$H \triangleq \begin{bmatrix} CB & CAB & \dots & CA^N B & \dots \\ CAB & CA^2 B & \dots & CA^{N+1} B & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ CA^N B & CA^{N+1} B & \dots & CA^{2N-1} B & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix} = [w^{(i+j-2)}(0)] .$$

It turns out that many qualitative input/output properties of Σ_1 are most easily described in terms of H .

It is well-known that there exist many minimal realizations $\{A, B, C\}$ of a given $G(s)$, $W(t)$, or H , but that they all may be recovered from one of them by the transformation group $\{A, B, C\} \xrightarrow{S} \{SAS^{-1}, SB, CS^{-1}\}$ with S an arbitrary invertible element of $R^{n \times n}$. The dimension of a minimal realization of a given transfer function is called the *McMillan degree*.

We will consider the following class of systems Σ_1 :

Definition 2: Σ_1 is said to be *completely symmetric* if $m=p$ and $H = H' \geq 0$.

*The infinite matrix H is said to be *nonnegative definite* (denoted by ≥ 0) if all its finite truncations are nonnegative definite, i.e. if $\sum_{i,j=0}^N z_i' CA^{i+j} B z_j \geq 0$ for all N and for all sequences $\{z_i\}_0^N$.

The following lemma gives a very useful alternative characterization of completely symmetric systems. Its proof, which is not germane to our purposes, is an immediate consequence of some known facts in realization theory and is left to the reader.*

Lemma 1: Σ is completely symmetric if and only if its transfer function $G(s) = C(Is-A)^{-1}B$ admits a realization $\{A_1, B_1, C_1\}$ with $A_1 = A_1'$ and $B_1 = C_1'$.

Thus Σ_1 is completely symmetric if and only if there exists a nonsingular $(n \times n)$ matrix S such that $SAS^{-1} = (SAS^{-1})'$ and $SB = (CS^{-1})'$. Completely symmetric systems have the property that the eigenvalues of A are all real. This is in fact also the case after applying symmetric feedback and it may be shown that Σ_1 is completely symmetric if $G(s) = G'(s)$ and if $A-BKC$ has real eigenvalues for all $K = K'$. Note also that Σ is completely symmetric if and only if its transfer function admits the partial fraction expansion $G(s) = \sum_{i=1}^k \frac{R_i}{s+\lambda_i}$ with $R_i = R_i' \geq 0$. If $m=p=1$ then Σ_1 is completely symmetric if and only if the poles and the zeros of the transfer function $G(s)$ are real and interlace, i.e. if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the poles and if z_1, z_2, \dots, z_r are the zeros of $G(s)$, then $r = n-1$, λ_1 and z_1 are real, and $\lambda_1 > z_1 > \lambda_2 > \dots > z_{n-1} > \lambda_n$. This pole-zero pattern is illustrated in Figure 2.

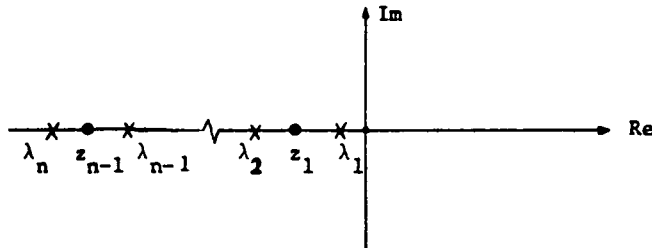


Figure 2: Typical pole/zero pattern of a completely symmetric system.

Completely symmetric systems are a natural generalization of *relaxation systems* (see Willems [1972]) which are completely symmetric systems which satisfy the additional stability requirement $\text{Re } \lambda[A] \leq 0$. Thus Σ_1 is a relaxation system if and only if its transfer function admits a realization $\{A_1, B_1, C_1\}$ with $A_1 = A_1' \leq 0$ and $B_1 = C_1'$. There are various other ways of defining a relaxation system. It may be

* The background material of realization theory used here may be found in Brockett [1970], Chapter 2, or Kalman [1969], Section 10.11.

shown that Σ_1 defines a relaxation system if and only if $H = H' \geq 0$ and $\sigma H = \sigma H' \leq 0$, where σH denotes the *shifted* Hankel matrix of Σ_1 , i.e., H with the first block row (or column) deleted. Alternatively, Σ_1 defines a relaxation system if and only if its impulse response $W(t) = Ce^{At}B$ is a *completely monotonic* function on $[0, \infty)$, i.e. $W(t) = W'(t)$ and $(-1)^k \frac{d^k}{dt^k} W(t) \geq 0$ for all $t \geq 0$ and $k = 0, 1, 2, \dots$. Relaxation systems play an important role in physics. They describe the response of various classes of systems such as R-C and R-L electrical networks, viscoelastic materials thermal systems, and chemical reactions.

We now state the main result of this section.

Theorem 1: Assume that Σ_1 is completely symmetric and that $K = K'$ almost surely.

Let $\overline{\lambda_{\max}} \triangleq \mathcal{E}\{\lambda_{\max}[A-BK]\}$. Then Σ is almost surely asymptotically stable if $\overline{\lambda_{\max}} < 0$.

Proof: The proof of Theorem 1 follows an argument due to Wazewski adapted to the case under consideration (as in Brockett [1970], Section 32, Exercise 6).

Since Σ_1 is completely symmetric, there exists a nonsingular matrix S such that $A_1 = SAS^{-1} = (SAS^{-1})' = A_1'$ and $B_1 = SB = (CS^{-1})' = C_1'$. Let $x_1 = Sx$. Then x_1 satisfies the equation:

$$\dot{x}_1 = (A_1 - B_1 K(t) B_1') x_1.$$

Let $V(x_1) = x_1' x_1$. Then along solutions of the above equation we have:

$$\dot{V}(x_1) = 2x_1' (A_1 - B_1 K(t) B_1') x_1,$$

which, since $A_1 - B_1 K(t) B_1'$ is symmetric, shows that:

$$\dot{V}(x_1) \leq 2\lambda_{\max}[A_1 - B_1 K(t) B_1'] V(x_1).$$

Since $A_1 - B_1 K(t) B_1'$ and $A - BK(t)C = S^{-1}(A_1 - B_1 K(t) B_1')S$ are similar matrices, this yields:

$$\dot{V}(x) \leq 2\lambda_{\max}[A - BK(t)C] V(x_1).$$

Thus

$$V(x_1(t)) \leq V(x_1(t_0)) \exp\left(2 \int_{t_0}^t \lambda_{\max}[A - BK(\tau)C] d\tau\right).$$

Finally by the ergodic hypothesis

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \lambda_{\max}[A - BK(\tau)C] d\tau = \overline{\lambda_{\max}}$$

is almost surely negative, which shows that $\lim_{t \rightarrow \infty} V(x_1(t)) = 0$ almost surely. Thus

$\lim_{t \rightarrow \infty} x_1(t) = S \lim_{t \rightarrow \infty} x(t) = 0$ almost surely, which proves the theorem. ■

Note: 1. Theorem 1 predicts stability if $\text{Re } \lambda(A) < 0$ and $K = K' \geq cI > 0$ almost surely. It then reduces to a special case of the multivariable circle criterion.

The major difficulty in applying Theorem 1 is that as a rule $\overline{\lambda_{\max}}$ will be difficult to compute from the distribution of K since $\lambda_{\max}[A-BKC]$ is a very nonlinear function K which does not even admit a general analytic expression. This difficulty may however be overcome in the important special case that there is only one stochastic gain in Σ :

Theorem 2: Assume that $m=p=1$ and that Σ_1 is completely symmetric with transfer function $g(s) = C(Is-A)^{-1}B$. Let z_1 be the largest zero of $g(s)$ and assume that $K(t)$ possesses the density function $p(K)$. Then Σ is almost surely asymptotically stable if:

$$\int_{z_1}^{\infty} \sigma p\left(-\frac{1}{g(\sigma)}\right) \frac{\partial g(\sigma)/\partial \sigma}{g^2(\sigma)} d\sigma > 0.$$

Proof: By Theorem 1 it suffices to prove that the integral in the theorem statement equals $-\overline{\lambda_{\max}}$. Consider therefore $\lambda_{\max}[A-BKC]$. Since the eigenvalues of $A-BKC$ are the poles of the system obtained after putting the constant feedback gain K around Σ_1 , it follows that these eigenvalues are the zeros of $1+Kg(s)$. Since the poles and zeros of $g(s)$ are real and interlacing it follows from a simple root-locus consideration that the maximum zero of $1+Kg(s)$ is a monotone decreasing function of K which varies from z_1 for $K = \infty$ to $+\infty$ for $K = -\infty$. The gain K and $\lambda_{\max}[A-BKC]$ are in fact related by $g(\lambda_{\max}[A-BKC]) = -\frac{1}{K}$. Thus, by a standard formula from probability theory we have that:

$$\mathcal{E}\{\lambda_{\max}[A-BKC]\} = \int_{z_1}^{\infty} \sigma p\left(-\frac{1}{g(\sigma)}\right) \left| \frac{\partial g(\sigma)/\partial \sigma}{g^2(\sigma)} \right| d\sigma$$

which yields the desired result. ■

Notes: 2. Figure 3 shows the behavior of the functions $g(\sigma)$, $-1/g(\sigma)$, and $\lambda_{\max}(A-BKC)$. The qualitative behavior of these functions is very well understood as a result of exhaustive analysis of R-C and R-L electrical networks (see, e.g., Guillemin [1957], Chapter 4).

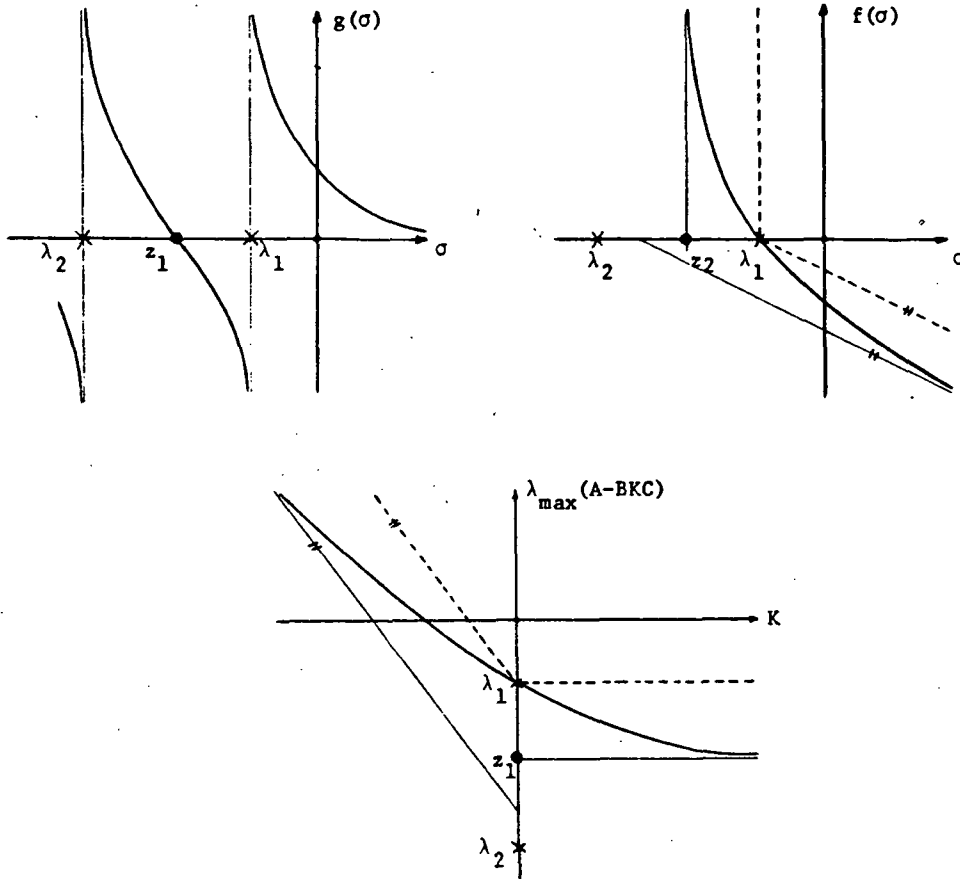


Figure 3: Sketch of $g(\sigma)$, $f(\sigma) = -\frac{1}{g(\sigma)}$, and $\lambda_{\max}(A-BKC)$

3. Theorem 2 indicates the destabilizing effect of the stochastic gain. To see this, let us assume (essentially without loss of generality) that $\mathcal{E}(K) = 0$. It may be shown that $\lambda_{\max}(A-BKC)$ is a strictly convex function of K which by Jensen's inequality (see Feller [1966], p. 151) implies that $\overline{\lambda_{\max}} \geq \lambda_1$ with equality holding if and only if $K = 0$ almost surely. Note also that Theorem 2 is easily extended to the case where K does not possess a density function.

4. Let $g(s) = \frac{q_{n-1}s^{n-1} + \dots + q_0}{s^n + p_{n-1}s^{n-1} + \dots + p_0}$ and let $\lambda_1, \dots, \lambda_n$ and z_1, \dots, z_{n-1} denote the poles and the zeros of $g(s)$. Thus $\lambda_1 > z_1 > \lambda_2 > \dots > z_{n-1} > \lambda_n$. Let $\lambda_{\max}(K) = \lambda_1(K) > \dots > \lambda_n(K)$ denote the zeros of $p(s) + Kq(s)$. From root-locus considerations it is easily seen

that $z_1 < \lambda_1(K) < \lambda_1(0) < \lambda_1(-K) < z_{i-1}$ for $K > 0$ and $i=1,2,\dots,n$ (where we have put $z_0 \triangleq \infty$ and $z_n \triangleq -\infty$). Since $-\sum_{i=1}^n \lambda_i + Kq_{n-1} = -\sum_{i=1}^n \lambda_i(K)$ we thus obtain the following upper bound for λ_{\max} (see Figure 3):

$$\lambda_{\max} \leq \begin{cases} \lambda_1 & \text{for } K \geq 0 \\ \lambda_1 - Kq_{n-1} & \text{for } K \leq 0 \end{cases}$$

This shows that Σ is almost surely asymptotically stable if:

$$\mathcal{E}\left\{\min\left[-\frac{\lambda_1}{q_{n-1}}, \frac{K-\lambda_1}{q_{n-1}}\right]\right\} > 0$$

which requires in particular that $\lambda_1 < 0$.

Examples: 1. If K is uniformly distributed between the limits K_- and K_+ then Σ is almost surely asymptotically stable if:

$$\frac{z_+}{g(z_+)} - \frac{z_-}{g(z_-)} + \int_{z_+}^{z_-} \frac{d\sigma}{g(\sigma)} > 0$$

where $z_+ \triangleq \lambda_{\max}(K_+)$ and $z_- \triangleq \lambda_{\max}(K_-)$. This inequality is easily verified directly from the graph of $f(\sigma) = -\frac{1}{g(\sigma)}$.

2. The limiting behavior of λ_{\max} as $K \rightarrow \pm \infty$ is given by (see Figure 3):

$$\lambda_{\max} \rightarrow \begin{cases} z_1 & \text{for } K \rightarrow \infty \\ -Kq_{n-1} + \alpha & \text{for } K \rightarrow -\infty \end{cases}$$

where $\alpha = \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} z_i = \frac{q_{n-2}}{q_{n-1}} - p_{n-1}$. Thus as K becomes more and more distributed at large absolute values we see that almost sure asymptotic stability results if:

$$z_1 P_+ + \left(\sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} z_i\right) P_- - q_{n-1} \int_{-\infty}^{\infty} K p(K) dK < 0$$

where $P_+ \triangleq P(K > 0)$ and $P_- \triangleq P(K < 0)$. For the uniformly distributed case studied in Example 1 with $K_+ > 0$ and $K_- < 0$ this condition requires

$$z_1 K_+ + \left(\sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} z_i\right) K_- + q_{n-1} \frac{K_-^2}{2} < 0$$

3. Consider the equation studied by Infante [1968], p. 11:

$$\dot{n} = \frac{f(t)-B}{\ell} n + \lambda c \quad ; \quad \dot{c} = \frac{B}{\ell} n - \lambda c$$

where $\beta, \ell, \lambda > 0$. This equation describes the kinetics of a simple nuclear reactor problem. It is easily seen that Theorem 2 applied to this case with

$$g(s) = \frac{1}{\ell} \frac{s+\lambda}{s(s+\frac{\beta}{\ell} + \lambda)} \quad \text{and } k(t) = -f(t).$$

Thus almost sure asymptotic stability results if:

$$\int_0^{\infty} (\sigma-\lambda) p\left(\frac{(\sigma-\lambda)(\ell\sigma+\beta)}{\sigma}\right) \left(\ell + \lambda\beta\frac{1}{\sigma^2}\right) d\sigma < 0$$

where $p(\cdot)$ denotes the density function of f .

1.2 A Frequency-Domain Stability Criterion

In this section we will derive another criterion for almost sure asymptotic stability of the system Σ . We first recall the definition of a positive real function:

Definition 3: Let $H(s)$ be a matrix of real rational functions of the complex variable s . It is said to be *positive real* if $H(s) + H'(\bar{s}) \geq 0$ for all $\text{Re } s \geq 0$, $s \neq$ poles of $H(s)$.

There exist various equivalent conditions for positive realness. Such conditions may be found in most books on electrical network synthesis (see, for example, Guillemin [1957], Chapter 1, or Newcomb [1966]). Positive real functions play a fundamental role in the theory of passive systems, particularly in the analysis and synthesis of electrical networks. They have recently also shown to be an essential tool for obtaining frequency-domain stability criteria for feedback systems. A time-domain condition for positive realness is given in the following lemma, the celebrated *Kalman-Yacubovich-Popov lemma*:

Lemma 2: Consider the minimal system:

$$\dot{z} = Fz + Gv; \quad w = Hz,$$

and let σ be a real number. Then $H(I(s-\sigma)-F)^{-1}G$ is positive real if and only if there exists a solution $O = O' > 0$ to the relations:

$$F'Q + QF \leq -2\sigma O;$$

$$OG = H'.$$

For a proof of Lemma 2 we refer the reader to Willems [1972].

The value of the above lemma in stability analysis lies in the fact that the quadratic form induced by the matrix Q yields a very suitable candidate for a Lyapunov function. It plays a crucial role in the following theorem which is the main result of this section:

Theorem 3: Let $m = p$. Then E is almost surely asymptotically stable if there exists a constant (n x n) matrix Λ and a real number σ such that:

$$(i) \quad \Lambda + \Lambda' > 0 ;$$

$$(ii) \quad F(s-\sigma) \triangleq G(s-\sigma)(I-\Lambda(s-\sigma)G(s-\sigma))^{-1} \text{ is positive real;}$$

$$\text{and} \quad (iii) \quad \mathcal{O}(\min[\sigma, \lambda_{\min}[(K+K')(\Lambda+\Lambda')^{-1}]]) > 0 .$$

Proof: We will assume that $(I-\Lambda CB)$ is invertible and that the McMillan degree of $F(s)$ is n . The general case may be resolved by a subsequent limiting argument which is left to the reader.

It is easily seen that $F(s)$ is the transfer function of the system:

$$\dot{z} = Az + B(v+\Lambda\dot{w}) ; \quad w = Cz ,$$

or

$$\dot{z} = (A+B(I-\Lambda CB)^{-1}\Lambda CA)z + B(I-\Lambda CB)^{-1}v ; \quad w = Cz .$$

This system is minimal since the McMillan degree of $F(s)$ is assumed to be n . Thus by condition (ii) and Lemma 2 there exists a matrix $Q = Q' > 0$ such that

$$[A+B(I-\Lambda CB)^{-1}\Lambda CA]'Q + Q[A+B(I-\Lambda CB)^{-1}\Lambda CA] \leq -2\sigma Q$$

and

$$QB(I-\Lambda CB)^{-1} = C'$$

Let S be an invertible (n x n) matrix such that $S'S = Q$ and let $x_1 = Sx$. The equation for x_1 is given by:

$$\dot{x}_1 = (A_1 - B_1 K(t) C_1) x_1$$

where $A_1 = SAS^{-1}$, $B_1 = SB$, and $C_1 = CS^{-1}$. Moreover, $B_1 = C_1'(I-\Lambda C_1 B_1)$ and $(A_1 + C_1' \Lambda C_1 A_1)' + (A_1 + C_1' \Lambda C_1 A_1) \leq -2\sigma I$. Consider now the derivative of $V(x_1) = x_1' x_1 + y_1' \Lambda y_1$, where $y_1 = C_1 x_1$, along solutions of the above differential equation. A simple calculation using the above relations shows that:

$$\dot{V}(x_1) \leq -2\sigma x_1' x_1 - 2y_1' K(t) y_1 = -2\sigma V(x_1) + 2y_1' (\sigma \Lambda - K(t)) y_1 .$$

Let $\lambda(t) = \lambda_{\min}[(K(t)+K'(t))(\Lambda+\Lambda')^{-1}]$ and let P be a nonsingular matrix such that $P'P = \Lambda + \Lambda'$. Since $\lambda(t) = \lambda_{\min}[P^{-1}(K(t)+K'(t))(P')^{-1}]$ it thus follows that $y_1'K(t)y_1 \geq \lambda(t)y_1'\Lambda y_1$ for all y_1 . Hence

$$\dot{V}(x_1) \leq -2\sigma V(x_1) + 2(\sigma - \lambda(t))y_1'\Lambda y_1.$$

We now distinguish two cases:

- (i) $\lambda(t) \geq \sigma$ which implies $\dot{V}(x_1) \leq -2\sigma V(x_1)$;
and (ii) $\lambda(t) \leq \sigma$ which, since $V(x_1) \geq y_1' \Lambda y_1$, implies:

$$\dot{V}(x_1) \leq -2\sigma V(x_1) + 2(\sigma - \lambda(t))V(x_1) = -2\lambda(t)V(x_1).$$

Hence

$$\dot{V}(x_1) \leq -2\min[\sigma, \lambda(t)]V(x_1)$$

and

$$V(x_1(t)) \leq V(x_1(t_0)) \exp(-2 \int_{t_0}^t \min[\lambda, \sigma(t)] dt).$$

By the ergodic hypothesis and condition (iii) this indeed implies that $\lim_{t \rightarrow \infty} V(x_1(t)) = 0$ almost surely. Thus $\lim_{t \rightarrow \infty} x_1(t) = S \lim_{t \rightarrow \infty} x(t) = 0$ almost surely, which proves the theorem. ■

Notes: 5. If $K + K' \geq \epsilon I > 0$ almost surely and if $G(s)$ is positive real then Theorem 2 predicts almost sure asymptotic stability by considering the limit $\sigma \rightarrow 0$ and $\Lambda \rightarrow 0$. In this sense Theorem 2 is thus a generalization of the circle criterion. The advantage of the theorem is that it allows the gain $K(t)$ to become negative provided however this is compensated by $K(t)$ being sufficiently positive at some other time.

One of the disadvantages of Theorem 3 is the inherent difficulty in verifying the average value condition from the distribution of K since $\lambda_{\min}\{(K+K')(\Lambda+\Lambda')^{-1}\}$ is a very nonlinear function of K . In the scalar case however one may resolve the various conditions in Theorem 3 much further. Thus we arrive at the following more explicit criterion for systems with a single stochastic parameter:

Theorem 4: Assume that $m = p = 1$ and let $g(s) = C(Is-A)^{-1}B = \frac{q_{n-1}s^{n-1} + \dots + q_0}{s^n + p_{n-1}s^{n-1} + \dots + p_0}$

denote the transfer function of Σ_1 . Then Σ is almost surely asymptotically stable if there exists a real constant β such that

- (1) $\mathcal{E}\{\min[\beta, K]\} > 0$;

- (ii) the poles of $G(s)$ lie in $\text{Re } s < -q_{n-1}\beta$;
 and (iii) the locus of $G(j\omega - q_{n-1}\beta)$, $-\infty < \omega < \infty$, does not encircle or intersect the closed disc centered on the negative real axis of the complex plane and passing through the origin and the point $-\frac{1}{\beta}$.

Proof: By Theorem 3 it suffices to show that there exists a constant $\lambda > 0$ such that $F(s-\sigma) = g(s-\sigma)(1-\lambda(s-\sigma)g(s-\sigma))^{-1}$ is positive real and $\mathcal{E}\{\min[\sigma, k]\} > 0$. Note that this implies $\sigma > 0$. Now $F(s-\sigma)$ is positive real if and only if $F^{-1}(s-\sigma) = \frac{1}{g(s-\sigma)} - \lambda(s-\sigma)$ is positive real. Since $F^{-1}(s-\sigma) = (\frac{1}{q_{n-1}} - \lambda)(s-\sigma) + \frac{r(s-\sigma)}{q(s-\sigma)}$ with $r(s)$ a polynomial of degree at most $(n-1)$ it follows that $\lambda \leq \frac{1}{q_{n-1}}$ and that $F^{-1}(s-\sigma)$ will be positive real for some λ if and only if it is positive real for $\lambda = \frac{1}{q_{n-1}}$, which is thus the optimal value of λ to consider. The condition $q_{n-1} > 0$ follows from the frequency domain condition (iii) as a result of the behavior of $g(j\omega - \sigma)$ for $\omega \rightarrow \infty$. Pick now $\sigma = \beta q_{n-1}$.

In order to complete the proof of the theorem it suffices to show that $F^{-1}(s-\sigma) = \frac{1}{g(s-q_{n-1}\beta)} - \frac{1}{q_{n-1}} s + \beta$ is positive real. By one of the test of positive realness this can be achieved by proving that $\text{Re } F^{-1}(s-\sigma)|_{s=j\omega} \geq 0$ and (since $F^{-1}(s-\sigma)$ has no more zeros than poles) that the roots of $q(s-\sigma)$ lie in $\text{Re } s < \sigma$. The real part condition comes down to asking $g(s-\sigma)|_{s=j\omega}$ to have the non-intersection property stated in condition (iii). By the non-encirclement condition the roots of $p(s-\sigma) + kq(s-\sigma)$ lie in $\text{Re } s < \sigma$ for $k > \beta$. By letting $k \rightarrow \infty$ this implies that the roots of $q(s-\sigma)$ lie indeed in $\text{Re } s \leq \sigma$. By the non-intersection property $g(j\omega - \sigma) \neq 0$ for $-\infty < \omega < \infty$ and we conclude that the roots of $q(s-\sigma)$ indeed lie in $\text{Re } s < \sigma$ as desired. ■

Notes: 6. It may be shown that conditions (ii) and (iii) of Theorem 4 will be verified for $\beta \leq \beta_1$ if they are verified for β_1 . Thus the optimal β to consider is the smallest number which satisfies condition (i) of the theorem.

7. If K has density function $p(K)$ then condition (i) of Theorem 4 requires that:

$$h(\beta) \triangleq \beta \int_{\beta}^{\infty} p(K) dK + \int_{-\infty}^{\beta} K p(K) dK > 0$$

Now $\frac{dh(\beta)}{d\beta} \geq 0$, $h(0) \geq 0$ and $h(\infty) = \mathcal{E}\{K\}$. Thus there exists a β such that $h(\beta) > 0$ if and only if $\mathcal{E}\{K\} > 0$, and if so, then there exists a β^* such that $h(\beta) > 0$ for $\beta > \beta^*$. Thus Theorem 4 will predict almost sure asymptotic stability of Σ

if $\mathcal{E}[K] > 0$, if the poles of $g(s)$ lie in $\text{Re } s < q_{n-1}\beta^*$ and if $g(j\omega - q_{n-1}\beta^*)$ satisfies the frequency domain condition of Theorem 4. This procedure lends itself very nicely to the graphical analysis illustrated in Figure 4.

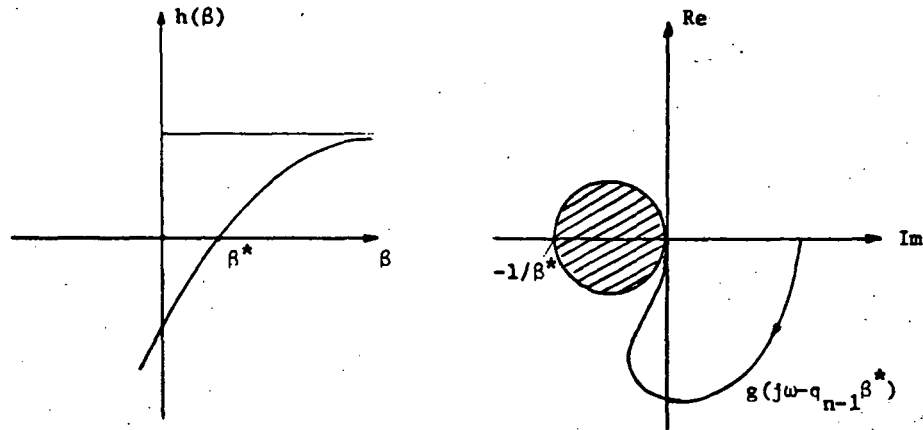


Figure 4: Illustrating the application of Theorem 4.

Examples: 4. Assume that K is uniformly distributed between K_- and K_+ with $K_- < 0$ and $K_+ + K_- \geq 0$. Then $\beta^* = K_+ - \sqrt{K_+^2 - K_-^2}$. Expressed in terms of the spread $\Delta K = K_+ - K_-$ and the mean $M = \frac{K_+ + K_-}{2}$ this yields $\beta^* = (\sqrt{\frac{\Delta K}{2}} - \sqrt{M})^2$ which in the range of interest $\frac{\Delta K}{2} \geq M \geq 0$ shows that β^* increases with ΔK for fixed M . This again indicates the destabilizing effect due to the uncertainty in K .

5. Let Σ_1 be a completely symmetric system as defined in Section 1.1. Then conditions (ii) and (iii) of Theorem 4 will be satisfied as long as $q_{n-1}\beta < -\lambda_1$ with λ_1 the largest pole of $g(s)$. The stability condition then becomes $\mathcal{E}\{\min[-\frac{\lambda_1}{q_{n-1}}, K]\} > 0$ which is similar to, but more conservative than, the condition obtained in Note 4. Thus Theorem 2 which only applies to completely symmetric systems gives a sharper stability estimate than Theorem 4 which applies to general systems.

2. ANALYSIS OF THE MEAN AND THE COVARIANCE EQUATIONS

This last section of the paper is concerned with the stability analysis of the mean and the covariance of the state of Σ where $K(t)$ is assumed to be a white stochastic process. For simplicity we will consider only the case in which the

process $K(t)$ is scalar valued, but we will treat the non-stationary case. If we denote the mean of $K(t)$ by $\bar{k}(t)$ and the variance by $\bar{\sigma}^2(t)$ then Σ is described by the stochastic differential equation:

$$\Sigma' : dx = (A - \bar{k}(t)bc)x dt + \bar{q}(t)bcx d\beta ,$$

where $A \in R^{n \times n}$, $b \in R^{n \times 1}$, $c \in R^{1 \times n}$, and β denotes a Wiener process with zero mean and unit covariance. This stochastic differential equation is to be interpreted in the sense of Ito and we will take it as the starting point of our analysis.

It is well-known that if $\bar{k}(t)$ and $\bar{q}(t)$ are sufficiently smooth (e.g., locally integrable) then for all given $x(t_0)$ there exists a unique solution to Σ' for $t \geq t_0$. Let $\mu(t) \triangleq E\{x(t)\}$, $\Gamma(t) \triangleq E\{x(t)x'(t)\}$, and $R(t) \triangleq E\{(x(t)-\mu(t))(x(t)-\mu(t))'\}$ denote respectively the mean, the second moment matrix, and the covariance matrix of $x(t)$. These are governed by the equations:

$$\dot{\mu} = (A - \bar{k}(t)bc)\mu ;$$

$$\dot{\Gamma} = (A - \bar{k}(t)bc)\Gamma + \Gamma(A - \bar{k}(t)bc)' + \bar{q}^2(t)bc\Gamma c'b' ;$$

$$\text{and} \quad R(t) = \Gamma(t) - \mu(t)\mu'(t),$$

with initial conditions $\mu(t_0) = x(t_0)$ and $\Gamma(t_0) = x(t_0)x'(t_0)$.

We will be concerned with the asymptotic properties of these variables. The relevant stochastic stability concepts are now defined:

Definition 4: Σ' is said to be *asymptotically stable in the mean*, in the *mean square*, or in the *covariance* if, respectively, $\lim_{t \rightarrow \infty} \mu(t) = 0$, $\lim_{t \rightarrow \infty} \Gamma(t) = 0$, or $\lim_{t \rightarrow \infty} R(t) = 0$ for all given initial conditions $x(t_0)$.

It is easily seen from the relations $\Gamma(t) = R(t) + \mu(t)\mu'(t)$ and $R(t) = R'(t) \geq 0$ that mean square asymptotic stability implies stability in the mean and in the covariance. The stability of the mean is a standard deterministic stability problem for which many criteria have been derived. These criteria involve the transfer function $g(s) = c(Is - A)^{-1}b$ and properties of $k(t)$ as, for example, its bounds (e.g. in the circle criterion: see Brockett [1970], Section 35), bounds on its derivative, or its periodicity. The stability of the differential equation which expresses the evolution of the second moment matrix $\Gamma(t)$ is much more intricate

to analyze and we will show how criteria like the multivariable circle criterion may be used. If $\bar{q}^2(t) = 0$ then its stability is equivalent to the stability of the mean equation, whereas if $\bar{q}^2(t) \neq 0$ then more stringent conditions will have to be imposed.

2.1 Multilinear System Theory

It is easy to see that if x_1 and x_2 are vectors which satisfy the linear equations:

$$\begin{aligned} \dot{x}_1 &= A_1(t)x_1 ; & x_1 &\in \mathbb{R}^{n_1}, \\ \text{and} \quad \dot{x}_2 &= A_2(t)x_2 ; & x_2 &\in \mathbb{R}^{n_2}, \end{aligned}$$

then the product $x_1 x_2'$ satisfies also a linear equation, namely:

$$\frac{d}{dt} x_1 x_2' = A_1(t)x_1 x_2' + x_1 x_2' A_2'(t).$$

By taking $x_1 = x_2$ we see that if x satisfies a linear equation, then so does xx' .

This idea generalizes from quadratic forms to homogeneous p -th degree forms. These facts have been known at least since Lyapunov's thesis, but they have to the present time been used very little in system theory. They may for example be exploited in the minimization of homogeneous performance measures of degree $p > 2$ for linear dynamical systems.

The above ideas may be used in setting up transfer functions for a class of bilinear systems. We will make some use of the Kronecker product denoted here by

\otimes . Thus the Kronecker product of $M \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{p \times q}$ is the element $M \otimes R \in \mathbb{R}^{np \times mq}$ defined by:

$$M \otimes R \triangleq \begin{bmatrix} m_{11}R & m_{12}R & \dots & m_{1q}R \\ m_{21}R & m_{22}R & \dots & m_{2q}R \\ \vdots & \vdots & \ddots & \vdots \\ m_{p1}R & m_{p2}R & \dots & m_{pq}R \end{bmatrix}$$

The main use of this notation is that if an $(n \times n)$ matrix O is written in lexicographic notation as the n^2 -vector

$$o_v = \text{col}(q_{11}, q_{12}, \dots, q_{1n}, \dots, q_{n1}, q_{n2}, \dots, q_{nn})$$

$$\text{then } (MQ)_v = (I \otimes M)o_v.$$

Consider now the following lemma:

Lemma 3: Let $\{A, b, c\}$ be a minimal realization of the transfer function $g(s) = c(Is-A)^{-1}b$. Then the differential equation:

$$\dot{Q} = AQ + QA + bv' + vb' ; \quad w = cQ ,$$

defines a minimal realization on the $\frac{n(n+1)}{2}$ dimensional space of symmetric $(n \times n)$ matrices of the transfer function:

$$g^{[2]}(s) = (c \otimes I + I \otimes c)(Is - I \otimes A - A \otimes I)^{-1}(b \otimes I + I \otimes b) .$$

We will not give a detailed proof of this lemma. The proof exploits the fact that the above matrix equation describes the bilinear system

$$\frac{d}{dt} xx' = Axx' + xx'A' + bux' + xu'b' ;$$

$$yx' = cxx'$$

where $\dot{x} = Ax + bu$; $y = cx$.

The dynamical system identified in the statement of Lemma 3 plays an important role in the analysis of the covariance equation under consideration. We know from this lemma that controllability and observability will be preserved. The poles of $g^{[2]}(s)$ are given by $\{\lambda_1(A) + \lambda_j(A)\}$, $1, j=1, \dots, n$. There appears to be no convenient general formula for deriving $g^{[2]}(s)$ from $g(s)$. In a specific case however, it is a relatively straightforward matter to calculate $g^{[2]}(s)$.

Example: 6. Let $\{A, b, c\}$ be the standard controllable representation (see Brockett [1970], p. 106) of $G(s) = \frac{1}{s^2 + as + b}$. Then

$$g^{[2]}(s) = - \frac{1}{s^3 + 3as^2 + (2a^2 + 4b)s + 4ab} \begin{bmatrix} 2(s+2a) & 4 \\ s(s+2a) & 2s \end{bmatrix} .$$

2.2 The Circle Criterion for the Covariance Equation

We now return to the covariance equation:

$$\dot{\Gamma} = (A - k(t)bc)\Gamma + \Gamma(A - k(t)bc)' + \bar{q}^2(t)bc\Gamma c'h'$$

which we model as the feedback system:

$$\begin{aligned}\Sigma'_1 : \dot{Q} &= A_0 + QA' + bv' + vb' + bw b' ; \quad y = cQ, \quad z = cQc', \\ \Sigma'_2 : v &= -\bar{k}(t)y, \quad w = -\bar{q}^2(t)z.\end{aligned}$$

It follows from Lemma 3 that Σ'_1 is completely controllable and completely observable.

Let

$$\tilde{G}(s) \triangleq \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix},$$

where $y(s) = G_{11}(s)v(s) + G_{12}(s)w(s)$

and $z(s) = G_{21}(s)v(s) + G_{22}(s)w(s)$,

denote the transfer function of Σ'_1 . It is easily calculated that $\tilde{G}(s)$ is given by:

$$\tilde{G}(s) = \begin{bmatrix} c \otimes I + I \otimes c \\ c \otimes c \end{bmatrix} (Is - A \otimes I - I \otimes A)^{-1} [b \otimes I + I \otimes b \mid b \otimes b]$$

Thus the stability of the covariance equation is equivalent to the stability of a deterministic feedback system with $(n+1)$ feedback loops, with transfer function $\tilde{G}(s)$ in the forward loop and gain matrix

$$F(t) = \begin{bmatrix} \bar{k}(t)I & n \\ 0 & -\bar{q}^2(t) \end{bmatrix}$$

in the feedback loop.

The multivariable circle criterion and its various generalizations is thus immediately applicable to this situation. We will illustrate this only in the simplest case. Let $\|\cdot\|$ denote some norm on R^{n+1} and let matrix norms be induced norms. The small loop gain theorem due to Zames [1966] thus leads to:

Theorem 5: Assume that $\operatorname{Re} \lambda[A] < 0$. Then Σ' is asymptotically stable in the mean square if:

$$\left(\sup_{-\infty < \omega < \infty} \|\tilde{G}(j\omega)\| \right) \left(\sup_{-\infty < t < \infty} \|F(t)\| \right) < 1.$$

Unfortunately it does not appear to be an easy matter to express the above criterion as direct conditions on the original transfer function $g(s)$ and the functions $\bar{k}(t)$ and $\bar{q}^2(t)$. In the case that $\bar{k}(t)$ or $\bar{q}^2(t)$ are time-invariant

however it is possible to obtain a criterion which is a great deal more specific:

Corollary 1: Assume that $\bar{k}(t) = k$ is constant. Then Σ' is asymptotically stable in the mean square if:

$$\left(\sup_{-\infty < t < \infty} \bar{q}^2(t) \right) \left(\int_0^{\infty} (ce^{(A-kbc)t} b)^2 dt \right) < 1 .$$

Proof: The equation for Γ may be modelled as the feedback system:

$$\begin{aligned} \dot{Q} &= (A-kbc)Q + Q(A-kbc)' + bw b' ; \quad z = cQc' , \\ w &= q^2(t)z . \end{aligned}$$

The first system has $(ce^{(A-kbc)t} b)^2$ as impulse response. Since this is always nonnegative it follows that its Fourier transform attains its maximum for $\omega = 0$. Since this maximum is given by $\int_0^{\infty} (ce^{(A-kbc)t} b)^2 dt$ we obtain the corollary by applying the circle criterion in the scalar case. ■

Corollary 2: Assume that $\bar{q}^2(t) = q^2$ is constant and let

$$\hat{G}(s) \triangleq (c \otimes I + I \otimes c)(Is - A \otimes I - I \otimes A - q^2(b \otimes b)(c \otimes c))^{-1}(b \otimes I + I \otimes b) .$$

then Σ' is asymptotically stable in the mean square if:

$$(1) \quad \bar{q}^2 \int_0^{\infty} (ce^{At} b)^2 dt < 1 ;$$

$$\text{and } (11) \left(\sup_{-\infty < t < \infty} \bar{k}(t) \right) \left(\sup_{-\infty < \omega < \infty} \|\hat{G}(j\omega)\| \right) < 1$$

Proof: The equation for Γ may be modelled as the feedback system:

$$\begin{aligned} \dot{Q} &= AQ + QA' + bcQc'b' + bv' + vb' ; \quad y = CQ , \\ v &= -k(t)y . \end{aligned}$$

The first system has $\hat{G}(s)$ as transfer function and is stable if condition (1) is satisfied. The corollary thus follows from the multivariable circle criterion (see Brockett [1970], Section 33). ■

Notes: 8. The conditions of Corollary 1 may be expressed in terms of frequency-domain data. They then lead to conditions very similar to the deterministic circle criterion (see Willems and Blankenship [1971]).

9. J.L. Willems [1972] has obtained a number of criteria for systems as the one studied here. His criteria which are in the vein of Corollary 1 are sharper and

more explicit than those studied here.

10. It is well-known that the circle criterion gives the best conditions which may be proven by means of a quadratic Lyapunov function. However in the case under consideration one can obtain results by using "linear" Lyapunov functions. Indeed, one may view the equation describing Γ as a differential equation on the space P of nonnegative definite symmetric $(n \times n)$ matrices. Restricting our attention to this subset of the vector space S of symmetric $(n \times n)$ matrices does not buy us anything as far as stability is concerned (i.e. stability on P is equivalent to stability on S). However it enhances the likelihood that a particular function will be definite and thus greatly enlarges the class of Lyapunov functions. For example the function $\text{Trace } [P\Gamma]$ with $P = P' > 0$ is positive definite on P but not on S . It hence defines a suitable Lyapunov function for studying the mean square stability question. This method is exploited in Willems [1972].

CONCLUSIONS

We have presented here a number of results on the stability of linear systems with stochastic coefficients. Two average value criteria for almost sure stability were derived and we showed how one may use deterministic stability results like the multivariable circle criterion in order to obtain mean square stability criteria in the case the stochastic parameters are white noise processes.

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FINITE GROUP HOMOMORPHIC SEQUENTIAL SYSTEMS

by

R.W. Brockett^{*} and Alan S. Willsky^{**}

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^{**}Fannie and John Hertz Foundation Fellow, Dept. of Aeronautics and Astronautics, M.I.T., Cambridge, Mass.

Abstract

Because many systems of practical interest fall outside the scope of linear theory it is desirable to enlarge as much as possible the class of system for which a complete structure theory is available. In this paper a class of finite state sequential systems evolving in groups is considered. The concepts of controllability, observability, minimality, realizability, and the isomorphism of minimal realizations are developed.

Results which are analogous to -- but differ in essential details from -- those of linear system theory are derived. These results are potentially useful in such diverse areas as algorithmic design and algebraic decoding.

1. Introduction

The purpose of this paper is to discuss certain questions related to the modeling of the input-output behavior of dynamical systems. We work in the context of systems with finite input, output, and state sets which admit group operations. The motivation for this study comes from a desire to understand better the key results in linear system theory (linear sequential machines included), and, more importantly, it comes from a desire to embrace in an analogous theory a broader class of input-output models than has here-to-fore been possible. Our results are potentially useful in optimizing the basic recursions occurring in certain elementary numerical processes, the mechanization of algebraic decoding procedures, etc.

This paper might be regarded as a contribution to the investigation of system theory in the context of universal algebras. It does not include the vector space results as a special case but it does shed new light on the previous proofs in that context, in that it makes clear which results depend only on the additive group structure inherent in a vector space. We have not worked for the weakest hypothesis for each individual theorem but rather have sought to place all theorems in a common framework -- one motivated by linear theory.

Thus, a number of the results and proofs have direct analogs in linear theory, and the proofs are presented to emphasize the universality of these arguments. That is, one should read these results keeping the following in mind. In the theory of algebra, there are a few basic isomorphism theorems for groups, rings, vector spaces, etc., and one

obtains the results in one setting from those in another simply by replacing the key words with their analogs - e.g. group for ring and normal subgroup for ideal. The results here indicate that the same type of universal structure and isomorphism results will hold in a system-theoretic framework.

One of the most difficult steps in constructing a realization of input-output maps is the state assignment problem. This step is crucial in the design of recursive algorithms, filters, etc. One of the essential features of our work is that we give a recipe for solving some problems of this type.

2. Finite Group Homomorphic Sequential Systems

Of course an empirical theory should avoid making assumptions which cannot be verified experimentally. However it is nonetheless useful to be able to anticipate the consequences of various assumptions about the internal mechanism of a phenomena under study, even if we are, in principle, incapable of verifying or denying the assumptions on the basis of experimentation. In this paper we want to investigate the properties of certain finite state systems which evolve in state spaces which admit a group structure and we verify in a constructive way the existence of this structure given the input-output data.

Specifically, we consider a class of dynamical models of the form

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

where the input, output, and state spaces are the finite groups

$\mathcal{U} = (U, \cdot)$, $\mathcal{Y} = (Y, *)$, $\mathcal{X} = (X, \circ)$, respectively. The maps $a : \mathcal{X} \rightarrow \mathcal{X}$,

$b : \mathcal{U} \rightarrow \mathcal{X}$ and $c : \mathcal{X} \rightarrow \mathcal{Y}$ are assumed to be group homomorphisms.

Invoking an analogy with linear sequential systems, which are a special case, we call this a finite group homomorphic sequential system.

This class of systems has many things in common with discrete time linear systems. The most obvious is the following result.

Theorem 1 : The input, initial state, and output of a finite group homomorphic sequential system

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

are related by

$$x(k) = b[u(k-1)] \circ a[b[u(k-2)]] \circ \dots \circ a^{k-1}[b[u(0)]] \circ a^k[x(0)]$$

$$\triangleq \left\{ \prod_{i=0}^{k-1} a^{k-i-1}[b[u(i)]] \right\} \circ a^k[x(0)]$$

$$y(k) = c[b[u(k-1)]] * c[a[b[u(k-2)]]] * \dots * c[a^{k-1}[b[u(0)]]] * c[a^k[x_0]]$$

$$\triangleq \left\{ \prod_{i=0}^{k-1} c[a^{k-i-1}[b[u(i)]]] \right\} * c[a^k[x(0)]]$$

where a^k denotes k compositions of a with itself.

Proof: This result follows directly from the system equations and the fact that a and c are homomorphisms.

3. Realizability Criteria

In this section we give necessary and sufficient conditions for an input-output map to have a sequential realization of the type under consideration here.

Recall that a sequence of linear maps of E^m into E^q is realizable as the weighting patterns of a finite dimensional discrete time linear system if and only if the sequence satisfies a linear recursion.

What we find here is that a sequence of homomorphisms of \mathcal{U} into \mathcal{Y} is realizable as the "weighting pattern" of a finite group homomorphic sequential system if and only if the sequence satisfies a homomorphic recursion.

Let $\mathcal{U} = (U, \cdot)$ and $\mathcal{Y} = (Y, *)$ be finite groups. We then define $F(\mathcal{U}, \mathcal{Y})$ to be the finite set of maps of \mathcal{U} into \mathcal{Y} . $F(\mathcal{U}, \mathcal{Y})$ is a semigroup under the operation

$$(fg)(u) \triangleq f(u) * g(u) \quad f, g \in F(\mathcal{U}, \mathcal{Y})$$

Suppose π is a homomorphism of $\mathcal{Y} \times \dots \times \mathcal{Y}$ (r factors) $\triangleq \mathcal{Y}^r$ into \mathcal{Y} . Then π naturally induces a homomorphism $\hat{\pi}$ of $F(\mathcal{U}, \mathcal{Y})^r$ into $F(\mathcal{U}, \mathcal{Y})$:

$$\hat{\pi}(A_1, \dots, A_r)(u) \triangleq \pi(A_1(u), \dots, A_r(u)) \quad \forall u \in \mathcal{U}, \quad A_1, \dots, A_r \in F(\mathcal{U}, \mathcal{Y})$$

Theorem 2¹: Let \mathcal{U} and \mathcal{Y} be finite groups. Given a sequence of group homomorphisms $T_i : \mathcal{U} \rightarrow \mathcal{Y}$, $i = 0, 1, 2, \dots$, there exists a finite group \mathcal{X} and group homomorphisms $a : \mathcal{X} \rightarrow \mathcal{X}$, $b : \mathcal{U} \rightarrow \mathcal{X}$, and $c : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$T_i(\cdot) = c[a^i[b(\cdot)]]$$

if and only if there is an integer $r > 0$ and a homomorphism

¹It has recently been pointed out to us that for the special case of abelian groups a realizability result is given in reference [6].

$$p : \mathcal{Y}^r \rightarrow \mathcal{Y}$$

such that for $i = 0, 1, 2, \dots$

$$\hat{p}(T_1, \dots, T_{i+r-1}) = T_{i+r}$$

Proof : (Sufficiency) Suppose such a homomorphism exists. We construct the analog of what has, in the context of linear system theory, been called the standard observable realization [1]. Consider the map of \mathcal{Y}^r into itself defined by

$$a : (x_1, x_2, \dots, x_{r-1}, x_r) \rightarrow (x_2, x_3, \dots, x_r, p(x_1, x_2, \dots, x_r))$$

This is clearly a homomorphism if p is. Now define b , taking \mathcal{U} into \mathcal{Y}^r by

$$b : u \rightarrow (T_0(u), T_1(u), \dots, T_{r-1}(u))$$

and again this is a homomorphism if each of the T 's is. Define c taking \mathcal{Y}^r into \mathcal{Y} according to

$$c : (y_1, y_2, \dots, y_r) \rightarrow y_1$$

This too is a homomorphism. We claim that $c[a^1[b(\cdot)]] = T_1(\cdot)$.

This is true because of the recursion given by \hat{p} :

$$c[b(\cdot)] = c(T_0(\cdot), T_1(\cdot), \dots, T_{r-1}(\cdot)) = T_0(\cdot)$$

$$\begin{aligned} c[a[b(\cdot)]] &= c(T_1(\cdot), T_2(\cdot), \dots, \hat{p}(T_0, T_1, \dots, T_{r-1})(\cdot)) \\ &= c(T_1(\cdot), T_2(\cdot), \dots, T_r(\cdot)) = T_1(\cdot) \end{aligned}$$

.....

$$c[a^{r-1}[b(\cdot)]] = c(T_{r-1}(\cdot), T_r(\cdot), \dots, T_{2r-2}(\cdot)) = T_{r-1}(\cdot)$$

The rest of the relations follow in a similar manner by applying the recursion.

(Necessity) Suppose that $T_1(\cdot) = c[a^1[b(\cdot)]]$ for some set of homomorphisms a, b , and c with $a: \mathcal{X} \rightarrow \mathcal{X}$ being defined on a finite group.

Since the set of all maps of \mathcal{X} into itself is a finite set, we see that

$a^r = a^k$ for some $r > k \geq 0$. Then $a^{r+m} = a^{k+m}$ for all $m \geq 0$. Then

defining p as the projection onto the $(k+1)$ st component of an r -tuple

$$p(y_0, \dots, y_{r-1}) = y_k$$

we see that

$$\hat{p}(T_1, T_{i+1}, \dots, T_{i+r-1})(\cdot) = T_{i+k}(\cdot) = c[a^{i+k}[b(\cdot)]] = c[a^{i+r}[b(\cdot)]] = T_{i+r}(\cdot)$$

We remark that the proof shows that the only sequences of homomorphisms $\{T_i\}$ which can be realized by a finite state system are those which are periodic after a finite number of terms (see figure 1). The next result shows that a is an automorphism if and only if there is no "tail."

Corollary : Under the hypotheses of Theorem 2, there exists a realization with a an automorphism if and only if $T_{k+l} = T_k$ for some l and all $k = 0, 1, 2, \dots$

Proof : This follows from the fact that a is an automorphism of a finite group if and only if a^k is the identity automorphism for some $k > 0$.

In automata theory, one usually considers systems described by maps of the form $f : U^* \rightarrow Y$ where U^* is the set of all finite strings of elements in U and $f(u_0, \dots, u_{n-1})$ is the output of the system at time n following the application of the input string u_0, \dots, u_{n-1} (in this order). One can then ask which f 's come from finite group homomorphic sequential systems.

Theorem 3 : Given finite groups $\mathcal{U} = (U, \cdot)$ and $\mathcal{Y} = (Y, *)$, and an input-output map $f : U^* \rightarrow Y$. This can be realized as a finite group homomorphic sequential system if and only if: $T_1 : \mathcal{U} \rightarrow \mathcal{Y}$, defined by

$$T_1(u) = f(u, \underbrace{e, \dots, e}_i)$$

i identity inputs

are homomorphisms satisfying the conditions of Theorem 2, and

$$f(u_0, \dots, u_n) = T_0(u_n) * T_1(u_{n-1}) * \dots * T_n(u_0)$$

Proof : The proof is a straightforward calculation. ■

Note that the second condition in Theorem 3 is equivalent to the following: if $\omega_1, \omega_2 \in U^*$ and the length of ω_2 is k , then

$$f(\omega_1, \omega_2) = f(\omega_2) * f(\omega_1, e^k)$$

where $e^k \in U^*$ is the string of k identity inputs.

For an input-output map f corresponding to a finite group homomorphic sequential systems, one should think of the map from U^r into Y^r given by

$$\begin{aligned} y_r &= f(u_0, \dots, u_{r-1}) = T_0(u_{r-1}) * T_1(u_{r-2}) * \dots * T_{r-1}(u_0) \\ y_{r+1} &= f(u_0, \dots, u_{r-1}, e) = T_1(u_{r-1}) * T_2(u_{r-2}) * \dots * T_r(u_0) \\ &\vdots \\ y_{2r-1} &= f(u_0, \dots, u_{r-1}, e^{r-1}) = T_{r-1}(u_{r-1}) * T_r(u_{r-2}) * \dots * T_{2r-2}(u_0) \end{aligned}$$

as being the analog of the map corresponding to the Hankel matrix. As will be shown, the number of elements in the image space of this map equals the number of states in the "minimal realization" just as the rank of the Hankel matrix determines the dimension of the state space of a minimal linear realization.

4. Controllability, Observability, and Minimal Systems

One of the crucial results in linear system theory is that a system is minimal if and only if it is controllable and observable and any two controllable and observable realizations of the same input-output map differ at most by a choice of basis for the state space. This result has a natural analog here but the analog of a related result, namely the fact that any input-output map which has a linear realization has a controllable and observable linear realization, fails. This means we must characterize all those systems which have controllable and observable realizations and this is done in Theorem 8 below. We note that finite dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension, whereas finite groups can have the same number of elements and not be isomorphic. Thus the state space isomorphism theorems are decidedly more interesting here.

We say that the homomorphic sequential system

$$x(k+1) = b[u(k)] \circ a[x(k)] ; y(k) = c[x(k)]$$

which evolves in the group $\mathcal{X} = (X, \circ)$ is controllable from $x_1 \in X$ if for any $x_2 \in X$ there exists a sequence of controls in the input group such that the state is driven from x_1 to x_2 by this sequence. The system is said to be controllable if it is controllable from all $x \in X$. Two

states $x_1, x_2 \in X$ are said to be indistinguishable if, given any input sequence, the corresponding output sequences from the initial states x_1 and x_2 are identical. Otherwise, x_1 and x_2 are said to be distinguishable, and an input sequence that yields different output sequences from x_1 and x_2 is said to distinguish between x_1 and x_2 . We call the system observable if any distinct pair of states are distinguishable.

Theorem 4 : Consider the finite group homomorphic sequential system

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

with state group $\mathcal{X} = (X, \circ)$. Let $e_x \in X$ be the identity in \mathcal{X} . Then the system is controllable if and only if it is controllable from e_x . The states x_1 and x_2 are distinguishable, if and only if the identity control sequence distinguishes between them. Also x_1 is indistinguishable from x_2 if and only if $x_1 x_2^{-1}$ is indistinguishable from e_x .

Proof : These results are obtained by straightforward calculations. ■

Thus, as in the case of linear systems, the test for controllability reduces to a test for controllability from the identity, and the test for observability to a test for indistinguishability from the identity.

The next theorem gives a formula for the set reachable from the identity and the set indistinguishable from the identity.

Theorem 5 : If the finite group homomorphic sequential system

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

evolves in a group $\mathcal{X} = (X, \circ)$ with n elements then the set of states reachable from the identity is

$$\mathcal{R} = \{b(u_1) \circ a[b(u_2)] \circ \dots \circ a^{n-1}[b(u_n)] \mid u_1, \dots, u_n \in U\}$$

$$\stackrel{\Delta}{=} b(U) \circ ab(U) \circ \dots \circ a^{n-1}b(U)$$

The set of states indistinguishable from the identity is

$$\mathcal{K} = \text{Ker } c(\cdot) \cap \text{Ker } c[a(\cdot)] \cap \dots \cap \text{Ker } c[a^{n-1}(\cdot)]$$

The set \mathcal{R} is not necessarily a group but \mathcal{K} is a normal subgroup of \mathcal{X} .

Proof : With respect to the reachable set, this result is immediate from the formula

$$x(k+1) = b(u(k)) \circ a[b(u(k-1))] \circ \dots \circ a^{k-1}[b(u(1))] \circ a^k[x(1)]$$

and the observation that because of the stationarity of the system, any state reachable from the identity is reachable along a trajectory that contains no state more than once and thus is of length less than or equal to n .

If the input sequence is a string of identity elements then the output sequence from the identity state is simply a string of identity elements in \mathcal{Y} . If the output from the state x is to be indistinguishable from this string then it must happen that

$$c(x) = c[a(x)] = \dots = c[a^{n-1}] = \text{identity}$$

Can it happen that this set of equalities holds but $c[a^p(x)] \neq \text{identity}$ for some $p \geq n$? Clearly not because for any x , $a^i(x) = a^j(x)$ for some $n \geq i > j \geq 0$ because there are only n elements in X . This means that for any x and any positive integer p we have $a^p(x) = a^k(x)$ with $0 \leq k \leq n-1$, where k , of course, depends on x and p . (Actually for $n \geq 2$, we can replace $n-1$ by $n-2$ in the expressions for \mathcal{R} and \mathcal{K} , but while this is easy to prove for \mathcal{R} , the result for \mathcal{K} is more cumbersome and we have thus omitted it).

To see that \mathcal{K} is a normal subgroup we need only observe that the map of \mathcal{X} into \mathcal{Y}^n defined by

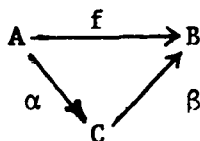
$$x \rightarrow (c(x), c[a(x)], \dots, c[a^{n-1}(x)])$$

is a homomorphism and \mathcal{K} is its kernel. That \mathcal{R} need not be a subgroup of \mathcal{X} will be shown by example later. ■

Corollary : Under the hypotheses of Theorem 5 the set \mathcal{R} is a subgroup if \mathcal{X} is an abelian group.

Proof : We need only note that for all $m \geq 0$, $a^m b(U)$ is a subgroup, and that the product of two subgroups of an abelian group is itself a subgroup. ■

We now recall some of the concepts of abstract realization theory ([2], Ch. 10). If A and B are sets and we have an input-output map $f : A \rightarrow B$, a factorization of f through a state set C is a pair of maps $\alpha : A \rightarrow C$ and $\beta : C \rightarrow B$ such that $f = \beta \circ \alpha$ - i.e. the following diagram commutes:



This factorization is canonical if α is onto and β is one-to-one.

In this case, the "size" of C is minimal in some sense. For instance if A , B , and C are vector spaces and f , α , and β are linear maps, and if \hat{C} , $\hat{\alpha}$, $\hat{\beta}$ is any other, not necessarily canonical, factorization, then $\dim C \leq \dim \hat{C}$. Also, if A , B , C , and \hat{C} are finite sets, with C corresponding to a canonical and \hat{C} to any other factorization, then $\text{card}(C) \leq \text{card}(\hat{C})$.

Suppose we have an input group $\mathcal{U} = (U, \cdot)$ an output group $\mathcal{Y} = (Y, *)$, and an input-output map $f : U^* \rightarrow Y$ that has at least one realization as a finite group homomorphic sequential system:

$$x(k+1) = b[u(k)] \circ a[x(k)], \quad y(k) = c[x(k)]$$

with finite state group $\mathcal{X} = (X, \circ)$. Suppose \mathcal{X} has n elements, and define $F : U^* \rightarrow Y^n$ by $F(u_0, \dots, u_k) = (f(u_0, \dots, u_k), f(u_0, \dots, u_k, e), \dots, f(u_0, \dots, u_k, e^{n-1}))$. We then have a factorization of F :

$$\begin{array}{ccc} U^* & \xrightarrow{F} & Y^n \\ & \searrow \mathcal{B} & \nearrow m \\ & \mathcal{X} & \end{array}$$

where

$$\begin{aligned} \mathcal{B}(u_0, \dots, u_k) &= b(u_k) \circ a b(u_{k-1}) \circ \dots \circ a^k b(u_0) \\ m(x) &= (c(x), ca(x), \dots, ca^{n-1}(x)) \end{aligned}$$

We immediately see that the above factorization is minimal if and only if the system is controllable and observable. In this case we say that the triple of homomorphisms (a, b, c) defines a minimal realization.

Another result of abstract realization theory is the following:
 given $f : A \rightarrow B$ and two canonical factorizations - that is two sets C and \hat{C} and corresponding maps $\alpha : A \rightarrow C$, $\hat{\alpha} : A \rightarrow \hat{C}$, both onto,
 and $\beta : C \rightarrow B$, $\hat{\beta} : \hat{C} \rightarrow B$, both one-to-one, such that $f : \beta \circ \alpha = \hat{\beta} \circ \hat{\alpha}$ - -
 then the two are equivalent, in that there exists a unique one-to-one and onto map $\gamma : C \rightarrow \hat{C}$, such that $\hat{\alpha} = \gamma \circ \alpha$ and $\beta = \hat{\beta} \circ \gamma$.

When we apply this result to the problem of finite group homomorphic sequential systems, we obtain stronger results, as in linear theory, because of the structure of the systems.

Theorem 6 : Suppose $\mathcal{U} = (U, \cdot)$ and $\mathcal{Y} = (Y, *)$ are finite groups, and $f : U^* \rightarrow Y$ is an input-output map that has two controllable and observable finite group homomorphic sequential realizations

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)] \quad (1)$$

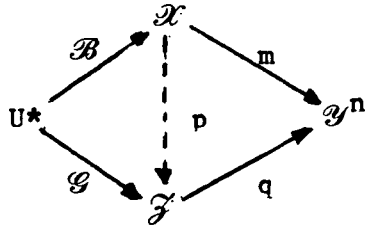
$$z(k+1) = g[u(k)] \circ f[z(k)] \quad ; \quad y(k) = h[z(k)] \quad (2)$$

where the system (1) evolves in a finite state group $\mathcal{X} = (X, \circ)$ and system (2) evolves in a finite state group $\mathcal{Z} = (Z, \circ)$. Then there exists a group isomorphism $p : \mathcal{X} \rightarrow \mathcal{Z}$ such that $f = pap^{-1}$, $g = pb$, and $h = cp^{-1}$. The two realizations are said to be conjugate.

Proof : Suppose the cardinality of \mathcal{X} is n . Then the same is true of \mathcal{Z} by the comments preceding the theorem. Let $F : U^* \rightarrow \mathcal{Y}^n$, $B : U^* \rightarrow \mathcal{X}$, and $m : \mathcal{X} \rightarrow \mathcal{Y}^n$ be as before, and define $\mathcal{G} : U^* \rightarrow \mathcal{Z}$ and $q : \mathcal{Z} \rightarrow \mathcal{Y}^n$ by

$$\begin{aligned} \mathcal{G}(u_0, \dots, u_k) &= g(u_k) \circ fg(u_{k-1}) \circ \dots \circ f^k_g(u_0) \\ q(z) &= (h(z), hf(z), \dots, hf^{n-1}(z)) \end{aligned}$$

Then, by controllability and observability we have two canonical factorizations of F and the commutative diagram



where p is the unique one to one and onto map such that the diagram remains commutative.

Let $x_1, x_2 \in X$. Then we have

$$q[p(x_1 \circ x_2)] = m(x_1 \circ x_2) = m(x_1) * m(x_2) = q[p(x_1)] * q[p(x_2)] = q[p(x_1) \circ p(x_2)]$$

Since q is one-to-one $p(x_1 \circ x_2) = p(x_1) \circ p(x_2)$. Thus p is an isomorphism. It is then a simple computation to arrive at the relation between (a, b, c) and (f, g, h) . ■

Note that in the theorem, the group structure of \mathcal{U} is never used, however the group structure of \mathcal{Y} and the fact that m and q are both one-to-one homomorphisms is used to show that p is an isomorphism. This lack of symmetry in the arguments is discussed in the next section.

As was mentioned in Theorem 5, \mathcal{R} - the set of states reachable from the identity - need not be a subgroup. Thus, given a finite group homomorphic sequential system, there need not exist a controllable system of this type with the same input-output description. In fact, one might expect that a homomorphic sequential system has a minimal realization as a homomorphic sequential system if and only if the set \mathcal{R} of states

reachable from e_x is, in any particular realization, a subgroup. The example below shows that this need not be the case. If \mathcal{R} is a subgroup, we can restrict our homomorphisms to \mathcal{R} , modulo the kernel of $(c, ca, \dots, ca^{n-1}) : \mathcal{X} \rightarrow \mathcal{Y}^n$, and thus construct a controllable and observable homomorphic realization (a simple check shows that one can redefine the homomorphisms in a well-defined manner after extracting the kernel -- therefore there always exists an observable homomorphic realization). Thus, for example, if there exists a homomorphic realization with an abelian state group, there exists a controllable and observable homomorphic realization.

An example will illustrate these ideas. The dihedral group, D_n , is a group of order $2n$ generated by two elements x and y which satisfy the relations

$$x^n = e, \quad y^2 = e \quad ; \quad xyx = y$$

where e is the group identity. The cyclic group of order n will be denoted as \mathbb{Z}_n , and its elements are $\{0, 1, \dots, n-1\}$. Consider the finite group homomorphic sequential system

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

where $\mathcal{U} = \mathcal{Y} = \mathbb{Z}_2$, $\mathcal{X} = D_4$, and a , b , and c are homomorphisms uniquely determined by

$$b(1) = y$$

$$a(x) = e, \quad a(y) = xy$$

$$c(x) = 0, \quad c(y) = 1$$

The set of states reachable from e may be shown to be

$$\mathcal{R} = \{e, y, xy, x^3\}$$

which is not a subgroup.

However if we compute the input-output homomorphisms $T_1 = ca^1b:Z_2 \rightarrow Z_2$, we find that

$$T_k = \text{identity for all } k \geq 0$$

Although the above nonminimal realization has an identity - reachable set which is not a group, there still exists a minimal homomorphic sequential system. In fact such a realization is found by taking $\mathcal{U} = \mathcal{X} = \mathcal{Y} = Z_2$ and $a = b = c = \text{identity}$. The reason we can find such a realization is that our original system is not observable. It is easy to see that there exists a controllable and observable homomorphic sequential realization of a given input-output map if and only if the identity-reachable set in any particular observable realization is a group. An example of an observable system for which \mathcal{R} is not a group is found by modifying the previous example. Let \mathcal{U} , \mathcal{X} , a , and b be as above, but let $\mathcal{Y} = \mathcal{X} = D_4$ and $c = \text{identity}$ (i.e. state output). This is observable, and \mathcal{R} is the same as before.

There are conditions under which \mathcal{R} is a subgroup, in which case we do have a controllable and observable homomorphic realization. The following theorem indicates one such condition.

Theorem 7 : Under the hypotheses of Theorem 5 the set \mathcal{R} of states reachable from the identity is a subgroup of \mathcal{X} if a is an automorphism.

Proof : The group of automorphisms of a finite group is itself a finite group with function composition as the group operation. Thus there exists a $k > 0$ such that

$$a^k = \text{identity automorphism}$$

From Theorem 1 we see that the set \mathcal{R} of states reachable from the identity can be written in the form

$$\begin{aligned} \mathcal{R} &= \bigcup_{m \geq 0} \prod_{i=0}^m a^{m-i} b(U) \\ &= \bigcup_{m \geq 1} [b(U) \circ ab(U) \circ \dots \circ a^{k-1} b(U)]^m \end{aligned}$$

where U is the input group and for $H \subset X$

$$H^m \triangleq \{h_1 \circ h_2 \circ \dots \circ h_m \mid h_i \in H\}$$

Thus if $x, y \in \mathcal{R}$, we have that $x \in [b(U) \circ ab(U) \circ \dots \circ a^{k-1} b(U)]^{m_1}$ and $y \in [b(U) \circ ab(U) \circ \dots \circ a^{k-1} b(U)]^{m_2}$ for some m_1 and m_2 . Then $xy \in [b(U) \circ ab(U) \circ \dots \circ a^{k-1} b(U)]^{m_1+m_2}$. We see that for all $n > 0$ $x^n \in \mathcal{R}$ if $x \in \mathcal{R}$. Since \mathcal{R} is a finite group, there exists an $N > 0$ such that $x^{-1} = x^N$. Therefore \mathcal{R} is a subgroup. █

The next theorem completely characterizes those sequences of input-output homomorphisms which have controllable and observable finite-group homomorphic sequential realizations. To do this, we must define what we mean by a free response of a system. If a system is given

in resursive form (as our first equation), a free response is the identity-input response of the system from some initial state. If the system is given in input-output form, it is the response to an input sequence which consists of the identity only, from some point onward, and where the response is observed from the point in time where the non-identity inputs stop. Thus we apply a (possibly) non-identity input up to time k and record the output from time $k+1$ on. Note that the set of free responses of an input-output map corresponds to the set of free responses of a homomorphic realization of that map started in a state reachable from the identity state. In what follows, free responses refer to the input-output system description. Note that we can consider the set of free responses as a subset of the infinite direct product group $\mathcal{Y} \times \mathcal{Y} \times \dots \times \mathcal{Y} \times \dots$.

Theorem 8: Let the sequence of homomorphisms $T_i : \mathcal{U} \rightarrow \mathcal{Y}$, $i=0,1,2,\dots$, with \mathcal{U} and \mathcal{Y} finite groups, satisfy the hypotheses of Theorem 2.

Then there exists a controllable and observable finite group homomorphic sequential realization if and only if the set of free responses form a subgroup of the infinite direct product group.

Proof : (Sufficiency) Let \mathcal{F} be the group of all free responses.

Let \mathcal{F}_n be defined as follows

$$\mathcal{F}_n = \left\{ (y_0, y_1, \dots, y_{n-1}) \in \mathcal{Y}^n \right. \quad \left. \begin{array}{l} y_0, y_1, \dots, y_{n-1} \text{ are the} \\ \text{first } n \text{ elements of a} \\ \text{free response } \in \mathcal{F} \end{array} \right\}$$

Obviously \mathcal{F}_n is a subgroup of \mathcal{Y}^n if \mathcal{F} is a subgroup of the infinite direct product.

Consider the standard observable realization given in the proof of Theorem 2. In that realization, the state space is $\mathcal{Y}^{\mathbb{I}}$, and it is easy to see that the set \mathcal{R} of states reachable from the identity is just $\mathcal{F}_{\mathbb{I}}$. Then, restricting our homomorphisms to $\mathcal{F}_{\mathbb{I}}$, we have a minimal homomorphic realization.

(Necessity) Suppose we have a minimal homomorphic realization of the T_i :

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

Since every state is reachable from the identity, the set of free responses in the input-output sense is identical to the set of free responses in the state space sense. Consider the map from \mathcal{X} into the infinite direct product group $\mathcal{Y} \times \mathcal{Y} \times \dots \times \mathcal{Y} \times \dots$ given by

$$x \rightarrow (c(x), ca(x), \dots, ca^k(x), \dots)$$

This is obviously a homomorphism, and its image is \mathcal{F} , which therefore must be a group.

Corollary : Under the hypothesis of Theorem 8, if \mathcal{F} is a group, \mathcal{F} is isomorphic to \mathcal{F}_n for some n .

Proof : Suppose a is the state transition homomorphism for a minimal realization. Then there exist $k > p > 0$ such that $a^k = a^p$, and then

$$(c(x), ca(x), \dots, ca^n(x), \dots) = (c(x), ca(x), \dots, ca^{k-1}(x), ca^p(x), \dots, ca^{k-1}(x), \dots)$$

and the isomorphism is obvious. Note that even if \mathcal{F} is not a group, there exists an n such that the elements of \mathcal{F} and \mathcal{F}_n are in one-to-one correspondence.

5. Some Comments on State Space Reduction

A number of questions were raised in the preceding sections. We have derived the standard observable realization - what about a "standard controllable realization" in the sense of reference [1]? The set of states indistinguishable from the identity is a (normal) subgroup - why isn't the set of states reachable from the identity a subgroup? In Theorem 6 we used the fact that m and q are homomorphisms - what about \mathcal{B} and \mathcal{G} ? We have seen that \mathcal{R} need not be a group, and for similar reasons \mathcal{B} and \mathcal{G} aren't homomorphisms and there is no standard controllable realization.

Note that these difficulties arise from the following consideration. Suppose we have a set of homomorphisms c_i , $i=1,2,\dots,m$ mapping a finite group \mathcal{X} into a finite group \mathcal{Y} . Then the "fan out" map taking \mathcal{X} into \mathcal{Y}^n

$$x \mapsto (c_1(x), \dots, c_n(x))$$

is always a homomorphism.

but the "fan in" map taking \mathcal{X}^n into \mathcal{Y}

$$(x_1, \dots, x_n) \mapsto c_1(x_1) \cdot c_2(x_2) \cdot \dots \cdot c_n(x_n)$$

need not be a homomorphism. (For example, the map of $\mathcal{X} \times \mathcal{X}$ into \mathcal{X} defined by group multiplication is typically not a homomorphism).

In the rest of this section, we will discuss these problems in some depth. We will also present some additional conditions which enable us to circumvent some of the difficulties.

Even if \mathcal{R} is a group, we cannot be sure that the map \mathcal{B} is a homomorphism. If \mathcal{X} has n elements, then the map \mathcal{B} defined by

$$\mathcal{B}: \mathcal{U} \times \dots \times \mathcal{U} \text{ (n times)} \rightarrow \mathcal{R}$$

$$\mathcal{B}(u_0, \dots, u_{n-1}) \stackrel{\Delta}{=} b(u_{n-1}) \circ ab(u_{n-2}) \circ \dots \circ a^{n-1}b(u_0)$$

is onto. We would like to investigate putting a semi-direct product structure on $\mathcal{U} \times \dots \times \mathcal{U}$ in order to make \mathcal{B} a homomorphism. We have the following necessary condition:

Theorem 9 : Consider a finite group homomorphic sequential system. If there exists a semidirect product structure on $\mathcal{U} \times \dots \times \mathcal{U}$ (n times) such that $\mathcal{B}: \mathcal{U} \times \dots \times \mathcal{U} \rightarrow \mathcal{R}$ is a homomorphism, then the set of states reachable from the identity in k steps is a group for all $k > 0$.

Proof : Choose $k \in \{1, \dots, n\}$. Consider the set of input strings

$$\mathcal{W}_k = \{(e_u^{n-k}, u_0, \dots, u_{k-1}) \mid u_0, \dots, u_{k-1} \in U\}$$

For any semidirect product structure on $\mathcal{U} \times \dots \times \mathcal{U}$, this is a subgroup. Thus $\mathcal{B}(\mathcal{W}_k)$ is a subgroup if \mathcal{B} is a homomorphism and $\mathcal{B}(\mathcal{W}_k)$ is just the set of elements reachable from the identity in k steps. For $k > n$ use Theorem 5. ■

We now modify the earlier example. We concern ourselves with the input-state side of the system only. Again let $\mathcal{U} = \mathbb{Z}_2$, $\mathcal{R} = D_4$, and let b be as before, but redefine a by

$$a(y) = xy, \quad a(xy) = y$$

It is easy to check that a is an automorphism of D_4 , and thus by Theorem 7 \mathcal{R} is a subgroup. However

$$\mathcal{B}(\mathcal{W}_2) = \{e, y, xy, x^3\}$$

which is not a group, and thus \mathcal{B} is not a homomorphism for any semi-direct product structure on $\mathcal{U} \times \dots \times \mathcal{U}$.

These examples illustrate an asymmetry in the theory. Unlike linear system theory - or even the abelian group case here, where it is clear that none of these difficulties appear - we do not have a naive duality theory without additional assumptions.

An assumption that avoids some of these difficulties is that of requiring a to be a normal endomorphism. A homomorphism f of a group \mathcal{G} into itself is called a normal endomorphism if for all $x, y \in \mathcal{G}$

$$xf(y)x^{-1} = f(xyx^{-1})$$

Theorem 10 : Consider the finite group homomorphic sequential system

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

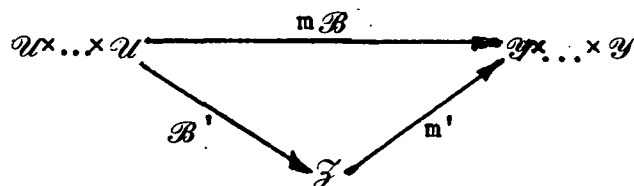
evolving in a finite group \mathcal{X} of order n . Suppose a is a normal endomorphism. Then there exists a semidirect product structure on $\mathcal{U} \times \dots \times \mathcal{U}$ (n times) such that the input-state map \mathcal{B} is a homomorphism, and thus the identity-reachable set \mathcal{R} is a subgroup.

Proof : Define the binary operation on $\mathcal{U} \times \dots \times \mathcal{U}$ (n times)

$$\begin{aligned} & (u_0, u_1, \dots, u_{n-1})(v_0, v_1, \dots, v_{n-1}) \stackrel{\Delta}{=} \\ & (v_1^{-1} \cdot v_2^{-1} \cdot \dots \cdot v_{n-1}^{-1} u_0 v_{n-1} \cdot \dots \cdot v_2 v_1 v_0, v_2^{-1} v_3^{-1} \cdot \dots \cdot v_{n-1}^{-1} u_1 v_{n-1} \cdot \dots \cdot v_3 v_2 v_1, \\ & \dots, v_{n-1}^{-1} u_{n-2} v_{n-1} v_{n-2}, u_{n-1} v_{n-1}) \end{aligned}$$

Direct computation verifies that this does define a semi-direct product structure on $\mathcal{U} \times \dots \times \mathcal{U}$ (n times), and another computation, using the fact that a is normal verifies that \mathcal{B} is a homomorphism. ■

Thus, in this case, we can reduce our system to a minimal homomorphic realization by first restricting the homomorphisms to \mathcal{R} and then taking \mathcal{R} modulo the kernel of m , the state-output map (see Theorem 6). We then have the following canonical factorization of the input-output map $m\mathcal{B}$



where \mathcal{Z} is the reduced state group, and \mathcal{B}' and m' are the reduced input-state and state-output homomorphisms, with \mathcal{B}' onto and m' one to one.

Another question arises in the case where \mathcal{R} is not a group. When this happens, we have $x_1, x_2 \in \mathcal{R}$ such that $x_1 \circ x_2 \notin \mathcal{R}$. Thus this particular group multiplication never occurs in the operation of the system and is irrelevant information. One can then ask whether or not we can redefine these irrelevant multiplications in such a manner as to make \mathcal{R} a group, while at the same time requiring that a, b , and c remain homomorphisms when restricted to \mathcal{R} . The example given previously shows that, at least in some cases, this can be done. Again let $\mathcal{U} = \mathcal{Y} = \mathbb{Z}_2$, $\mathcal{X} = D_4$ with a, b, c defined by $b(1) = y$; $a(x) = e$,

$a(y) = xy$; $c(x) = 0$, $c(y) = 1$. We saw that

$$\mathcal{R} = \{e, y, xy, x^3\}$$

The superfluous multiplications are $(xy) \circ y$, $(xy) \circ x^3$, $x^3 \circ y$, and $x^3 \circ x^3$. If we define these as follows

$$\begin{aligned} (xy) \circ y &\triangleq x^3 & x^3 \circ y &\triangleq xy \\ (xy) \circ x^3 &\triangleq y & x^3 \circ x^3 &\triangleq e \end{aligned}$$

then \mathcal{R} is the Klein-4 group, and it is easy to check that a , b , and c are still homomorphisms. In fact, since the Klein-4 group is abelian, a is a normal endomorphism, and we can reduce our system as described above.

6. Conclusions

In this paper we have considered a broader class of input-output relations than those found in linear system theory and have derived results analogous to some of the more crucial properties of linear systems. In particular, we have considered dynamical systems of the form

$$x(k+1) = b[u(k)] \circ a[x(k)] \quad ; \quad y(k) = c[x(k)]$$

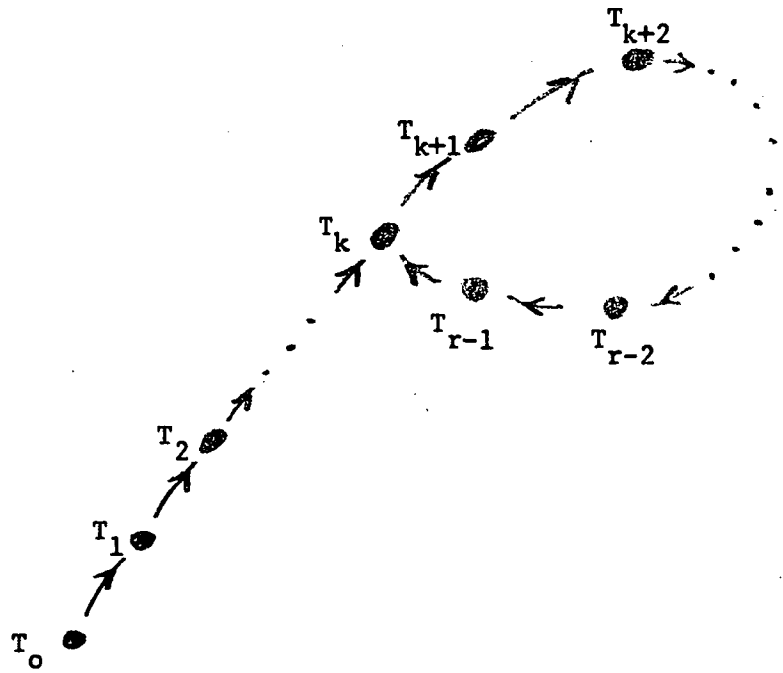
where the input, state, and output spaces are finite groups, and a , b , and c are homomorphisms. The concepts of controllability, observability, and minimality are developed, and conditions for the realization of an input-output map by such a system are given. As in the linear case, the equivalence of any two minimal homomorphic realizations is established.

In addition, several problems, all directly or indirectly related to duality, arise in considering this broader class of systems. These are discussed, and it is shown that an additional assumption removes these problems.

The analogy with linear theory has by no means been completely exploited. Concepts such as transform theory have not been considered at all. Also, extensions of some of these results to infinite group problems can be made, possibly making contact with the study of dynamical systems on topological groups [7].

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Illustrating the Realizability Condition

Figure 1

Lie Theory and Control Systems Defined on Spheres[†]

R.W. Brockett*

Abstract

We show in this paper that in constructing a theory for the most elementary class of control problems defined on spheres, some results from Lie theory play a natural role. In particular to understand controllability, optimal control, and certain properties of stochastic equations, Lie theoretic ideas are needed. The framework considered here is probably the most natural departure from the usual linear system/vector space problems which have dominated the control systems literature. For this reason our results are compared with those previously available for the finite dimensional vector space case.

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*Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. 02138.

1. Introduction

Specific results about control systems whose state spaces are spheres have been useful in understanding problems in energy conversion, controlled rigid body dynamics, etc. Some examples are mentioned in our earlier paper [1]. Here we work out in more detail, and in greater generality, the theory for a class of problems of this type and compare out results with the case where the state space is a vector space. To carry out this program requires some results from Lie theory, Lie groups acting on spheres, etc. There has been no attempt here to discuss the most general setting in which techniques which we use are applicable. Instead we have taken the sphere problems as a model and have studied a range of control-theoretic questions in that setting. A number of possible generalizations will be apparent.

To begin with we mention some well known facts about linear system theory. We do this to make the paper a little more accessible to those not familiar with control problems and to sensitize the reader to certain issues important in control. For a more complete account and references to the literature one can consult [2] for the deterministic results and [3] for the stochastic results.

Linear system theory deals with the pair of equations

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ; \quad y(t) = Cx(t) \quad (1.1)$$

where \dot{x} denotes a time derivative. It is assumed that $x(t) \in \mathcal{K}^n$, $u(t) \in \mathcal{K}^m$ and $y(t) \in \mathcal{K}^p$. For simplicity we take A, B, C to be constant matrices. One calls u the control, x the state and y the output. The theory of linear

system is extensive but for our present purposes we point out only the following five results.

i) (1.1) is said to be controllable if for every x_0 and x_1 in \mathcal{R}^n and every $t_1 > 0$ there exists a piecewise continuous control $u(\cdot)$ such that if $x(0) = x_0$ then $x(t_1) = x_1$. A necessary and sufficient condition for controllability is that $\text{Rank}(B, AB, \dots, A^{n-1}B) = n$ where \cdot indicates a column partition.

ii) (1.1) is said to be observable if for every $x_1 \neq x_2$ and every $t_1 > 0$ the outputs corresponding to x_1 and x_2 differ on the interval $[0, t_1]$. A necessary and sufficient condition for observability is that $\text{rank}(C; CA; \dots, CA^{n-1}) = n$ where \cdot indicates a row partition.

iii) If (1.1) is controllable then for every given x_0 and x_1 in \mathcal{R}^n and every $t_1 > 0$ there exists a piecewise continuous control u defined on $[0, t_1]$ which transfers the state from x_0 at $t = 0$ to x_1 at $t = t_1$ and minimizes

$$\eta(t) = \int_0^{t_1} u'(t)u(t)dt \quad (1.2)$$

relative to all other piecewise continuous controls which accomplish the same transfer.

iv) If there exists a linear feedback control law $u = Fx$ such that $\dot{x} = (A+BF)x$ has a null solution which is asymptotically stable then there exists a control law $u = Kx$ such that $\lim_{t \rightarrow \infty} x(t) = 0$ and the functional

$$\eta = \int_0^{\infty} u'(t)u(t) + y'(t)y(t)dt$$

is minimized by setting $u(t) = Kx(t)$.

v) If (1.1) is controllable and if the differential equation $\dot{x} = Ax$ is asymptotically stable then the associated stochastic equation (for notation see [3]).

$$dx(t) = Ax(t)dt + Bd\omega(t) \quad (1.3)$$

has a unique invariant Gaussian measure which has zero mean and variance Q satisfying

$$QA + A'Q = -BB' \quad (1.4)$$

In this paper we establish analogs for each of these results for systems of the type

$$\dot{x}(t) = (A + \sum_{i=1}^m u_i(t)B_i)x(t) \quad ; \quad y(t) = Cx(t) \quad (1.5)$$

where A, B_1, B_2, \dots, B_m are skew symmetric matrices and the system can be thought of as evolving on the sphere $||x(t)|| = ||x(0)||$.

One significant point in the linear theory is that the matrix B is generally not invertible and cases for which it is invertible are so infrequent as to be virtually without interest. If B is invertible then by an appropriate choice of basis equation (1.1) becomes

$$\dot{x}(t) = Ax(t) + u(t) \quad (1.6)$$

and controllability is automatic. Moreover, in this case problems iii) and iv) are easily reduced to variational problems of the classical type

$$\eta = \int_0^{t_1} L(x, \dot{x}) dt \quad (1.7)$$

with L quadratic in x and \dot{x} and $L_{\dot{x}\dot{x}}$ positive definite. Control theory works with the more general "degenerate" case where $L_{\dot{x}\dot{x}}$ is only nonnegative definite but certain constraints are in effect. If the above integral is

thought of as the action integral in a mechanics problem then the case treated in control theory allows for the possibility of certain zero masses provided there are appropriate linear constraints between position and velocity. It can also be thought of as a limiting case of an unconstrained dynamical problem where certain masses and associated energies go to infinity. This second interpretation is generally more useful. Remarks of the same type apply to equation (1.3) where existence of a smooth transition density is well known if B is invertible whereas the same is true, but for rather more subtle reasons, if we assume controllability instead of invertibility of B .

2. Controllability

One of the main areas of applicability of Lie theory in control has been that of determining the set of points reachable along solution curves of $\dot{x}(t) = f(x(t), u(t), t)$ for the set of all piecewise continuous controls $u(\cdot)$. For studies of this kind see references [4-10]. If the control equations are of the form

$$\dot{x}(t) = (A + \sum_{i=1}^m u_i(t) B_i) x(t) \quad ; \quad x(t) \in \mathbb{R}^n \quad (2.1)$$

then the system typically evolves on a manifold in \mathbb{R}^n . The determination of the set of points reachable from a given point x_0 can be accomplished by the determination of the set of matrices reachable from the identity for the matrix equation

$$\dot{X}(t) = (A + \sum_{i=1}^m u_i(t) B_i) X(t) \quad ; \quad X(0) = I \quad (2.2)$$

and then letting this set act on x_0 via ordinary matrix-vector multiplication. Equation (2.2) can be thought of as defining a control problem on a matrix Lie group. The question of determining what matrices are reachable from the identity along solutions of (2.2) has been the subject of a number of papers [1, 7-10]. Following Jurdjevic and Sussmann, we term systems of the form of (2.2) right invariant. This is appropriate because the vector fields defined on the $GL(n)$ by the right side of (2.2) are invariant under the translation defined by right multiplication with an element of $GL(n)$. We will say that equation (2.2) is controllable on a group \mathcal{G} if any two points in \mathcal{G} can be joined by a solution curve generated by some piecewise continuous control $u(\cdot)$.

Suppose that A and B_1, B_2, \dots, B_m are all skew symmetric. Then regardless of the choice of u the solutions of equation (2.1) remain on the sphere defined by $||x(t)|| = ||x(0)||$. We will say that the system (2.1) is controllable on the sphere if any two points on the sphere be joined by a solution curve generated by some piecewise continuous curve $u(\cdot)$. Phrased another way, the system is controllable if the set of matrices reachable from the identity along solutions of (2.2) act transitively on S^{n-1} . From earlier results [10] we know that since the motion is confined to a subgroup of $SO(n)$ the set of matrices reachable from I is the matrix Lie group consisting of all the matrices which can be expressed as products of the form $\exp H_1 \exp H_2 \dots \exp H_n$ where H_1, H_2, \dots, H_n belong to the Lie algebra generated by A, B_1, B_2, \dots, B_n .

Now of course the orthogonal group $SO(n)$ acts transitively on S^{n-1} so that if the algebra generated by A, B_1, B_2, \dots, B_m is the full set of skew symmetric matrices then the system (2.1) is controllable on S^{n-1} . However there are certain subgroups of $SO(n)$ which act transitively on S^{n-1} as well. The real compact forms of the classical Lie groups are all candidates. The results are well known [11] but we repeat them here. For example, it is clear that both the full unitary group and the special unitary group of dimension n act transitively on the set of complex n -vectors whose Hermetian length is one. But this set is just a set of vectors with components $(x_i + \sqrt{-1} y_i)$ such that

$$\sum_{i=1}^n (x_i^2 + y_i^2) = 1 \quad (2.3)$$

which is a $2n-1$ dimensional sphere. Thus by defining the realification [12]

of the unitary algebras by the Lie algebra homomorphism

$$B \mapsto \begin{bmatrix} \operatorname{Re} B & \operatorname{Im} B \\ -\operatorname{Im} B & \operatorname{Re} B \end{bmatrix} \quad (2.4)$$

we obtain a set of real matrices whose associated group acts transitively on S^{2n-1} . The real compact form of C_n is the intersection of special unitary group and the symplectic groups. Naturally this representation is in terms of matrices of even dimension so that they can act on even dimensional complex vectors only. Thus, by analogy with the unitary case, the real compact form of C_n acts on the sphere of dimension S^{4n-1} . This action is known to be transitive and of course we can add to the algebra real multiples of $\sqrt{-1} I$ to get the "full quaternion-unitary group" which acts transitively as well.

These four cases, each valid for all integer n , together with three particular ones account for all possibilities. The particular cases may be explained as follows. The exceptional algebra G_2 admits a 7 dimensional skew-symmetric representation whose exponential acts transitively on S^6 . The spin representation of $SO(7)$ is 8 dimensional and it acts transitively on S^7 . The spin representation of $SO(9)$ is 16 dimensional and it acts transitively on S^{15} . With this explanation we can state the following result.

Theorem 1: Let A, B_1, \dots, B_m be a collection of n by n skew symmetric matrices. The control system

$$\dot{x}(t) = (A + \sum_{i=1}^m u_i(t) B_i) x(t) \quad (2.5)$$

is controllable on S^{n-1} if the algebra generated by A, B_1, B_2, \dots, B_m is

- i) $SO(n)$ for $n \equiv 0 \pmod{2}$
- ii) $SO(n)$ or the realification of $SU(n/2)$ or $U(n)$ for $n \equiv 1 \pmod{2}$
- iii) The realification of $Sp(n/2)$ for $n \equiv 1 \pmod{4}$
- iv) G_2 if $n = 6$, $Spin(8)$ if $n = 7$ or $Spin(16)$ if $n = 15$

Moreover, if the Lie algebra is not one of these cases the system (2.8) is not controllable.

If the system is not controllable on S^{n-1} it is sometimes of interest to compute exactly what points can be reached from a given initial state. The determination of what points belong to this set is facilitated by a knowledge of the structure of the representation defined by the matrices in the algebra generated by A, B_1, B_2, \dots, B_m . If this representation is not irreducible then its reduction is clearly the first step in the determination of the reachable set. The properties of the irreducible pieces may reveal the form of the reachable set in a straightforward way. For example, if the evolution equation can be decomposed as

$$\dot{x} = [I \otimes A^1 + A^2 \otimes I + \sum_{i=1}^m u_i (I \otimes B_i^1 + B_i^2 \otimes I)]x(t) \quad (2.6)$$

then the Kronecker product of the reachable group for

$$\dot{X}(t) = (A^1 + \sum_{i=1}^m u_i(t) B_i^1)X(t) \quad (2.7)$$

and the reachable group for

$$\dot{X}(t) = (A^2 + \sum_{i=1}^m u_i(t) B_i^2)X(t) \quad (2.8)$$

contains the reachable group for equation (2.2). The reachable group will not, in general, simply be the Kronecker product of the reachable groups unless the effects of the u 's are decoupled.

For the linear evolution equation (1.1) it happens that if it is possible to transfer any state to any other state then this transfer can be done in arbitrarily small time. This is not the case for systems defined by

equation (2.1). Jurdjevic and Sussmann [9] give an example of a system defined on S^2 which is controllable but certain transfers cannot be made in less than 1 unit of time. Thus if (1.1) is controllable on S^n the strongest statement we can make on the basis of the present analysis is that for t_1 sufficiently large every state can be transferred to every other state in t_1 units of time. Estimates on this time have not yet been worked out.

In the vector space case controllability is closely related to the concept of observability as mentioned in the introduction. In the present setting this is not the case at all. We say that the system

$$\dot{x}(t) = (A + \sum_{i=1}^m u_i(t)B_i)x(t) \quad ; \quad y(t) = Cx(t) \quad (2.9)$$

is observable on S^{n-1} if no two distinct initial states on S^{n-1} give rise to the same response y for all controls $u(\cdot)$. The following theorem gives a necessary and sufficient condition for observability.

Theorem 2: Let A, B_1, B_2, \dots, B_m be a collection of skew symmetric matrices and let c be a unit vector. The control system

$$\dot{x}(t) = (A + \sum_{i=1}^m u_i(t)B_i)x(t) \quad ; \quad y(t) = cx(t)$$

is observable on S^{n-1} if and only if the set of matrices $\{A, B_1, B_2, \dots, B_m, cc'\}$ are irreducible.

For a proof of this theorem and more general results of this type see [13].

3. Optimal Control

Consider again the evolution equation (2.2) defined on matrix group \mathcal{G} . Let there be given a time $t_1 > 0$ and boundary conditions of the form $X(0) = X_0$; $X(t_1) = X_1$. Suppose that in addition there is given a functional which is of the action type

$$\eta_1 = \frac{1}{2} \int_{t_0}^{t_1} \sum_{i=1}^m u_i^2(t) dt \quad (3.1)$$

as opposed to the geodesic type

$$\eta_2 = \int_{t_0}^{t_1} \left(\sum_{i=1}^m u_i^2(t) \right)^{1/2} dt \quad (3.2)$$

Our problem is to determine if there exists a control $u(\cdot)$ such that the boundary conditions are met and the given functional is minimized and, if such a control exists, to characterize it. Just as with controllability, there is an obvious connection between problems defined on a group and problems defined on a manifold on which that group acts. This would no longer be the case if η depended on x in a general way.

We will use the formalism of the maximum principle of Pontryagin [14] rather than the calculus of variations to attack this problem because it handles the degeneracy which is built into the problem in a natural way. Applied to the present problem, Pontryagin's maximum principle asserts that if $u(\cdot)$ is an optimizing control then there exists a matrix P such that

$$\dot{P}(t) = -A'P(t) - \sum_{i=1}^m u_i(t) B_i' P(t) \quad (3.3)$$

and H_j defined by

$$H(P, X, u) = \langle P, AX \rangle + \sum_{i=1}^m u_i \langle P, B_i X \rangle + \sum_{i=1}^m \frac{1}{2} u_i^2 \quad (3.4)$$

is minimized with respect to u by the optimal control. Thus we have the optimal control given by

$$u_i(t) = \langle -P(t), B_i X(t) \rangle \quad (3.5)$$

This choice of u gives a pair of differential equations with split boundary conditions

$$\frac{d}{dt} \begin{bmatrix} X(t) \\ P(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \begin{bmatrix} X(t) \\ P(t) \end{bmatrix} - \sum_{i=1}^m \langle P, B_i X \rangle \begin{bmatrix} B_i & 0 \\ 0 & -B_i' \end{bmatrix} \begin{bmatrix} X(t) \\ P(t) \end{bmatrix} \quad (3.6)$$

The problem can be reduced to a single quadratic equation with split boundary conditions by introducing $K = XP'$. An easy calculation shows that

$$\dot{K}(t) = AK(t) - K(t)A' - \sum_{i=1}^m \langle B_i', K(t) \rangle (B_i K(t) - K(t)B_i') \quad (3.7)$$

So far everything is valid for an arbitrary subgroup of $GL(n)$. If

A, B_1, B_2, \dots, B_m are self contragredient then a simplification occurs.

In that case any solution of the differential equation for P can be expressed in terms of a solution of the differential equation for X with nonsingular boundary conditions; i.e. $P(t) = NX(t)M$ for some constant matrices M and N . Specializing to the skew symmetric case gives the following result.

Theorem 4: Suppose that A, B_1, B_2, \dots, B_m are skew symmetric n by n matrices and suppose that there exists a piecewise continuous control $u(\cdot)$ which transfers the state of the matrix system

$$\dot{X}(t) = (A + \sum_{i=1}^m u_i(t) B_i) X(t) \quad (3.8)$$

from X_0 at $t = 0$ to X_1 at $t = t_1 > 0$. Then there exists constant matrices M and N such that the solution of

$$\dot{X}(t) = (A + \sum_{i=1}^m \langle B_i, X(t) M X'(t) N \rangle B_i) X(t) \quad ; \quad X(0) = X_0 \quad (3.9)$$

passes through X_1 at $t = t_1$. Moreover, there exists one such pair M, N which minimizes η_1 relative to any other continuous $u(\cdot)$ which steers the system to X_1 from X_0 in the same period of time.

Proof: That there exists an optimal control follows from theorem 6 of Cesari [15]. The rest follows from the maximum principle as discussed above.

There is an alternative point of view available for these problems which makes a little closer contact with both physics and Lie theory but which is not so useful here. Consider the right-invariant control equation in $SO(n)$ with control Ω

$$\dot{X}(t) = \Omega(t)X(t) \quad ; \quad X(0) = X_0 \quad (3.10)$$

Let the problem be to pick Ω in the space of skew symmetric matrices such that $X(t_1) = X_1$ and the trace form

$$\eta = \int_0^{t_1} \text{tr} (\Omega^{-1} \Omega)^2 dt \quad (3.11)$$

is minimized. Elementary variational arguments with due regard for the admissibility of variations lead to the Euler equation

$$\dot{\Omega} = \Omega \Omega \Omega^{-1} - \Omega^{-1} \Omega \Omega \quad (3.12)$$

In $SO(3)$ this matrix equation is equivalent to the familiar Euler equations for a rigid body

$$I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3 \quad (3.13)$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_1 \omega_3$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2$$

which, after all, come from minimizing the action integral on $SO(3)$.

(Note that the kinetic energy of a rigid body can be expressed by the trace form $(\det I) \text{tr}(I^{-1} \Omega)^2$ where I is the usual inertia tensor.

See [2] page 64. Incidentally, this also serves to define the degree of difficulty of actually solving the control problem mentioned above. Since it is well known that the solution of the Euler equations generally involves elliptic functions, the solution of the optimal control problems cannot be expressed in terms of elementary functions except in special cases.

By far the simplest special case on $SO(n)$ occurs when η_1 is the negative of the integral of the Killing form. That is given $X(0)$ and $X(1)$ and given the evolution equation

$$\dot{X}(t) = \sum_{i=1}^{n(n-1)/2} u_i(t) B_i X(t) ; \quad X \in SO(n) \quad (3.14)$$

where $B_i = -B_i'$ and for all i and j

$$\langle B_i, B_j \rangle = \text{tr } B_i B_j' = \delta_{ij} \quad (3.15)$$

one finds that the optimal trajectory is

$$X(t) = e^{\Omega t} X(0) \quad (3.16)$$

where Ω is the solution of $e^{\Omega} = X(1)X^{-1}(0)$ which has the smallest Frobenius norm.

We turn now to applying the above results to the problem of optimizing trajectories on spheres. Note that trajectories on spheres can be optimized for fixed end points by solving an associated right invariant group problem and then picking the minimizing element in the group for

transferring x_0 to x_1 . The following theorem expresses this.

Theorem 5: Let A, B_1, B_2, \dots, B_m be skew symmetric matrices. Suppose that the system

$$\dot{x}(t) = (A + \sum_{i=1}^n u_i(t)B_i)x(t) \quad (3.17)$$

is controllable on S^n . Then given a sufficiently large time $t_1 > 0$ and given points x_0 and x_1 in S^{n-1} , there exists a control which transfers the system from x_0 at $t = 0$ to x_1 at $t = t_1$ and minimizes

$$\eta = \int_0^{t_1} u'(t)u(t)dt \quad (3.18)$$

Moreover, there exists a matrix K_0 such that the optimal control is given by $u_i(t) = \langle K(t), B_i \rangle$ where K is defined by the matrix differential equation

$$\dot{K}(t) = [A, K(t)] + \sum_{i=1}^m \langle K(t), B_i \rangle [K(t), B_i] ; K(0) = K_0 \quad (3.19)$$

We complete this section on optimal control with a result of the type which plays a major role in linear system theory in connection with the regulator problem.

Theorem 6: Let A and B be n by n skew symmetric matrices and consider the system

$$\dot{x}(t) = Ax(t) + u(t)Bx(t) \quad (3.20)$$

Let a be a unit vector in the null space of A such that A and $Baa'B'$ are a pair of matrices which act irreducibly on the orthogonal complement of the one dimensional subspace defined by a . Then the control law $u(t) = a'Bx(t)$ steers the system from any initial state $x_0 \neq -a$ to a and minimizes the integral

$$\eta = \int_0^{\infty} u^2(t) + [a'Bx(t)]^2 dt$$

relative to any other continuous control $u(\cdot)$.

Proof: We can write η as

$$\eta = \int_0^{\infty} u^2(t) - 2a'\dot{x}(t) + [a'Bx(t)]^2 dt + 2a'x(t) \Big|_0^{\infty}$$

since $Aa = 0$ we have

$$\eta = \int_0^{\infty} (u(t) - a'Bx(t))^2 dt + 2a'x(t) \Big|_0^{\infty}$$

Thus if the control law $u(t) = a'Bx(t)$ actually drives the state x to a then it is optimal. However, observing that $a'x(t)$ has a derivative along the given solution which is equal to $-[a'Bx(t)]^2$, we see by LaSalle's theorem (see e.g. [2]) that the solution $x = a$ can fail to be stable if and only if $a'Be^{At}x$ vanishes identically for some $x \neq \pm a$.

By looking at the derivatives at $t = 0$ we see that this can happen if and only if $(Ba, ABa, \dots, A^{n-1}Ba)$ fails to span the orthogonal complement of the one dimensional subspace defined by a .

4. Stochastic Differential Equations

We consider now a third aspect of control theory on spheres. This has to do with the analog of property (v) mentioned in the introduction. What we show is that controllability implies the existence of a unique invariant measure for a stochastic equation on S^{n-1} . We use Ito notation for stochastic differential equations. Wong [3] can be consulted for an explanation of both the mathematics and the notation.

Let w_1, w_2, \dots, w_m denote independent Wiener (Brownian motion) processes of unity variance. In giving a precise meaning to differential equations in which something like "white noise" appears K. Ito [16] invented what has proven to be a very successful calculus in which the standard differentiation rule is significantly modified insofar as differentials of Wiener processes are concerned. In this calculus $dw_i dw_j = \delta_{ij} dt$, a first order term; $dw_i dt$, and $(dt)^2$ are both higher than first order. We discuss the implication of this in one important special case. If x and y are vectors satisfying the Ito differential equations

$$dx(t) = Ax(t)dt + Bx(t)dw(t) \quad (4.1)$$

$$dy(t) = Fy(t)dt + Gy(t)dw(t) \quad (4.2)$$

Then $z(t) = x(t)y'(t)$ satisfies the Ito equation

$$dz(t) = (Az(t) + z(t)F' + Bz(t)G')dt + (Bz(t) + z(t)G')dw \quad (4.3)$$

The only other fact we need about Ito equations concerns the associated mean equation. If x and y satisfy equations (4.1) and (4.2) then

$\bar{x}(t) = \mathcal{E}x(t)$ and $\bar{y}(t) = \mathcal{E}y(t)$ satisfy the ordinary differential equation

$$\frac{d}{dt} \bar{x}(t) = A\bar{x}(t) \quad (4.4)$$

$$\frac{d}{dt} \bar{y}(t) = F\bar{y}(t) \quad (4.5)$$

We will see that these two results permit the derivation of equations for all moments and imply that the moment equations are decoupled from each other.

Recall that the number of linearly independent degree p forms in n variables is given by

$$N(n,p) = \binom{n+p-1}{p} \quad (4.6)$$

We can therefore associate with each n tuple (x_1, x_2, \dots, x_n) a $N(n,p)$ -tuple $x^{[p]} = (x_1^p, \sqrt{p} x_1^{p-1} x_2, \dots, x_n^p)$ where the coefficients are chosen in such a way as to validate the equality

$$||x^{[p]}||^2 = ||x||^{2p} \quad (4.7)$$

It is clear that if x satisfies an ordinary differential equation which is linear, say

$$\frac{d}{dt} x(t) = Ax(t) \quad (4.8)$$

then $x^{[p]}$ also satisfies a linear differential equation

$$\frac{d}{dt} x^{[p]}(t) = A^{[p]} x(t) \quad (4.9)$$

We regard this as a definition of $A^{[p]}$. It is related to the classical idea of an induced representation. Of course if there are controls present a similar set of equations follow; i.e. equation (2.1) implies

$$\frac{d}{dt} x^{[p]}(t) = A^{[p]} x^{[p]}(t) + \sum_{i=1}^m u_i(t) B_i^{[p]} x^{[p]}(t) \quad (4.10)$$

Similar remarks hold for stochastic equations of the type under consideration here, provided suitable allowance is made for the Ito calculus. Associated with the Ito equation

$$dx(t) = Ax(t)dt + \sum_{i=1}^m B_i x(t)dw_i \quad (4.11)$$

is the family of equations

$$dx^{[p]}(t) = \left(\left(A - \sum_{i=1}^m \frac{1}{2} B_i^2 \right) [p] + \sum_{i=1}^m \frac{1}{2} (B_i^{[p]})^2 \right) x^{[p]}(t) dt + \sum_{i=1}^m B_i^{[p]} x^{[p]}(t) dw_i . \quad (4.12)$$

The derivation of this is a straightforward exercise using the properties of dw_i outlined above. Finally, we have the moment equations associated with (4.11)

$$\frac{d}{dt} \bar{x}^{[p]}(t) = \left(\left(A - \sum_{i=1}^m \frac{1}{2} B_i^2 \right) [p] + \sum_{i=1}^m \frac{1}{2} (B_i^{[p]})^2 \right) \bar{x}^{[p]}(t) \quad (4.13)$$

where $\bar{x}^{[p]}(t) = \mathbb{E} x^{[p]}(t)$. Compare with reference 17.

In terms of the Ito calculus when can the matrix stochastic equation

$$dX(t) = AX(t)dt + \sum_{i=1}^m dw_i(t) B_i X(t) \quad (4.14)$$

be thought of as evolving the orthogonal group? This will be the case when the associated vector equation (4.11) evolves on the sphere defined by $||x(t)|| = ||x(0)||$ for all $x(0)$. Using the facts outlined above we see that $d(x'x) = 0$ if and only if for all i

$$B_i = -B_i', \quad A - \sum_{i=1}^m \frac{1}{2} B_i^2 = -\left(A - \sum_{i=1}^m \frac{1}{2} B_i^2 \right)', \quad (4.15)$$

Thus these are the conditions under which equation (4.14) evolves in the orthogonal group and the conditions under which (4.11) evolves on the sphere.

It is apparent that the measure associated with the uniform density on the sphere is an invariant measure for the process defined by equation (4.11). Since the area of the $(n-1)$ -sphere is $2\pi^{n/2}/\Gamma(n/2)$ the uniform density is

$$\rho_0(x) = \Gamma(n/2)/2\pi^{n/2} \quad (4.16)$$

The corresponding values of the odd moments are zero by symmetry but the even moments are not. The following theorem claims that all the moments approach the moments associated with a uniform distribution if we have controllability. Incidentally, equation (4.13) provides a means for actually computing the moments for all time in terms of their values at $t = 0$.

Theorem 7: Suppose that A, B_1, B_2, \dots, B_m are all skew symmetric and suppose that

$$\dot{x}(t) = (A + \sum_{i=1}^m u_i(t) B_i) x(t) \quad (4.17)$$

is controllable on S^{n-1} . Then the solution of the Ito differential equation defined on the sphere by

$$dx(t) = (A + \sum_{i=1}^m \frac{1}{2} B_i^2) x(t) dt + \sum_{i=1}^m B_i x(t) dw_i \quad (4.18)$$

is such that all moments approach the moments associated with a uniform distribution on the $n-1$ sphere as t approaches infinity.

Proof: First of all, note the shift in notation from (4.11) to (4.18).

In (4.11) $A - \frac{1}{2} \sum B_i^2$ is playing the role played by A alone here. It is not difficult to show that because A, B_1, B_2, \dots, B_m are skew symmetric it follows that $A^{[p]}, B_1^{[p]}, B_2^{[p]}, \dots, B_m^{[p]}$ are also skew symmetric. A second observation concerns stability. If $A = -A'$ and $B_i = -B_i'$ then all solutions of the ordinary differential equation

$$\dot{x}(t) = (A + \sum_{i=1}^m \frac{1}{2} B_i^2) x(t) \quad (4.19)$$

are bounded. Moreover, each solution approaches zero as t approaches infinity provided $B_i e^{At} x$ does not vanish identically for any $x \neq 0$ and there will exist nonzero vectors such that $B_i e^{At} x$ vanishes identically if and only if A and B_i can be put in the form

$$\theta' A \theta = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \theta' B_1 \theta = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.20)$$

To prove the first of these facts we notice that since $A = -A'$

$$\frac{d}{dt} ||x(t)||^2 = - \sum_{i=1}^m ||B_i x(t)||^2 \quad (4.21)$$

Thus by LaSalle's theorem (see e.g. [2]) the solution either goes to zero or else there is a solution along which $||B_i x(t)||$ vanishes identically for all i . That solution would have to be of the form $e^{At} x_0$. As for the conditions on A and B_i , they follow from considering the subspace of vectors such that $B_i e^{At} x$ vanishes, together with its orthogonal complement, making use of the skew symmetry of A, B_1, B_2, \dots, B_m .

Clearly controllability implies that all solutions of the mean equation approach zero as t approaches infinity because controllable systems cannot be decomposed as indicated. As for the higher moments, we must distinguish between the even and odd cases. For the odd cases if there is a decomposition then controllability of the equation (4.17) is clearly impossible. For the even moments, we have in view of the identity $||x^{[p]}||^2 = ||x||^{2p}$, a decomposition of the type given by equation (4.20) but with the zero block in B_i being one dimensional. The one dimensional subspace defines the steady state value of the even moments. On the orthogonal complement the equation (4.18) is asymptotically stable. These remarks are related to some well known properties of orthogonal representations of Lie algebras.

As is well known, the moments $x^{[p]}$ are related to the spherical harmonics in a direct way. Thus by working with equation (4.13) it is possible to obtain a full solution to the Fokker-Plank equation associated with the Ito equation (4.18). The interpretation of the moments in terms of spherical harmonics also allows one to establish some qualitative features of the probability density. In particular its smoothness and convergence to the steady state can be easily studied.

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